FORMATION SUBGROUPS OF FINITE SOLUBLE GROUPS

by

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Preface

I declare that this dissertation, entitled "Formation subgroups of finite soluble groups", is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other University, and that no part of it has already been, or is being currently, submitted for any such degree, diploma, or other qualification. I further declare that with the exception of passages where specific reference is made to the work of others and of results which are described as "well-known", this dissertation is entirely my own original work.

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CONTENTS

Chapter One

INTRODUCTION

Historically this dissertation has its roots in a series of papers by P. Hall ([10] - [13]) which are basic to the study of soluble groups and which have been the cornerstone of many later developments in the subject. Hall showed that a group is soluble if and only if it possesses a Sylow p-complement for each prime p dividing its order, and he investigated the Sylow systems of a soluble groups, a Sylow system being the intersection lattice of a complete set of Sylow p-complements. A group is nilpotent if and only if it has a unique Sylow system and then the normalizer of this system is the whole group. In general, the normalizer of a Sylow system is nilpotent, and its size in relation to the size of the whole group provides a measure of how close that group comes to being nilpotent. A system normalizer covers the central chief factors of a group and avoids the eccentric chief factors, thus providing a connection between the normal and the Sylow structure of a group. The set of all system normalizers forms a characteristic conjugacy class of a group.

More recently, Carter has shown in [4] that a soluble group has another important characteristic conjugacy class of nilpotent groups which are characterized by their property of being self-normalizing; we shall call them the Carter subgroups of a soluble group. Carter's detailed investigation in [5] showed that these two classes of nilpotent subgroups are intimately connected; a Carter subgroup always contains a system normalizer and vice versa, and for special classes of soluble group precise information is obtained about the way a system normalizer is embedded in a Carter subgroup. In 1963 Gaschütz revealed in [8] his elegant theory of formations, and was able to show by closer scrutiny of Carter's methods that if $\mathfrak{F}$ is a saturated formation every finite soluble group has a canonical conjugacy class of $\mathfrak{F}$-subgroups which behave much like Carter subgroups and which in fact coincide with Carter subgroups when $\mathfrak{F}$ is the class of nilpotent groups. For reasons we shall explain in chapter five we call the canonical subgroups of Gaschütz the $\mathfrak{F}$ - covering subgroups of a group. The far-reaching generalization of Carter subgroups obtained from Gaschütz's theory of formations seemed strongly to indicate the possibility of a corresponding extension of Hall's theory of system normalizers. The search for this extension has been largely successful, and the attendant discoveries provide the main theme of this work. We report on a new canonical conjugacy class of subgroups called the f - normalizers of a soluble group, and show that they display in a generalized form many of the established properties of system normalizers; here f denotes a formation function defining the local formation $\mathfrak{F}$, and when $f(p) = 1$ for all p $\mathfrak{F}$ is the class of nilpotent groups and the f - normalizers become simply the system normalizers.

In chapter two a discussion of notation and terminology is followed by a short summary of elementary results about formations; we also define the concept of an integrated formation function which plays such an important part in the extension of results for system normalizers to the general situation. Section 2.3 brings in the closely related concepts of an f-normal (f-abnormal) maximal subgroup and an f-central (f-eccentric) chief factor which specialize to the usual concepts of normal and central when $f(p) = 1$ for all p. With the help of these ideas we are able to prove in Theorem 2.4.7, the culminating theorem of the chapter, two necessary and sufficient conditions for a group G to belong to the formation $\mathfrak{F}$; they are that every maximal subgroup of G should be f-normal, and that every chief factor of G should be f-central. In fact we prove these and subsequent results in a more general form for a soluble group L embedded as a normal subgroup in an arbitrary finite group G, but to avoid the cumbersome terminology involved in stating our results "relative to G" we limit ourselves to the case $L = G$ in this introductory discussion.

In chapter three we have a preliminary encounter with the main theme by studying the p-theory of f-normalizers which is distinct from but often analogous to the general theory of f-normalizers; many of the results of this chapter prove useful in the sequel. We show that for every p-local formation $\mathfrak{F}_p$ defined by $f(p)$ a finite soluble group has a characteristic conjugacy class of $\mathfrak{F}_p$-subgroups called $f(p)$ - normalizers (Definition 3.1.3 and Theorem

3.3.6). The $f(p)$ - normalizers are homomorphism-invariant (Theorem 3.3.4) and cover all but the f-eccentric p-chief factors which they avoid (Theorem 3.3.1). The notion of an $f(p)$-critical maximal subgroup enables us to show that the $f(p)$ - normalizers of a group G can be joined to G by an $f(p)$-critical maximal chain of subgroups and to show with the help of Lemma 3.1.6 that they can be characterized abstractly as the minimal members of the set of all subgroups which can be joined to G by an f-abnormal p-maximal chain.

The analogy with chapter four is now evident. For there we show that the f - normalizers of a finite soluble group G form a homomorphism-invariant characteristic conjugacy class of subgroups which cover the f-central chief factors and avoid the f-eccentric chief factors of G ; if f is integrated, they belong to $\mathfrak{F}$ and are precisely the minimal members of the set of subgroups which may be joined to the whole group by an f-abnormal maximal chain. The concept of an f-critical maximal subgroup, that is an f-abnormal maximal subgroup supplementing the Fitting subgroup, is again important. Every group G which is not in the local formation $\mathfrak{F}$ has an f-critical maximal subgroup M ; M eliminates one f-eccentric chief factor of G and preserves the remaining chief factors in a chief series of G . The f - normalizers of G are the terminal members of the f-critical maximal chains. An arbitrary maximal subgroup of G contains an f - normalizer of G if and only if it is f-abnormal; in fact an f - normalizer of an f-abnormal maximal subgroup of G always contains an f - normalizer of G. Moreover, if X is a subgroup of G supplementing $F(G)$, then an f - normalizer of X has the form $X \cap D$ where D is a suitable f - normalizer of G. The intersection of the f - normalizers of G is the f-hypercentre of G and their join is G itself. In section 4.5 we give examples to show that many of our theorems about f - normalizers turn out to be false when the restriction that f is integrated is lifted, but we also prove that some of the results can be saved if we impose the alternative condition that f is S-closed. We end chapter four by proving a theorem from which it follows that under certain conditions an absolute f - normalizer of a relative f - normalizer of a normal subgroup of G is an absolute f - normalizer of G.

Chapter five is devoted to a discussion of two new characterizations of Gaschütz's $\mathfrak{F}$ - covering subgroups. The first of these enables us to generalize the concept of an $\mathfrak{F}$ - covering subgroup to the "relative" situation and so we make it a definition from which to develop afresh some of the known properties of $\mathfrak{F}$ - covering subgroups as well as to derive some new theorems. Apart from the intrinsic interest of an alternative approach, our main justification for doing this is that having discussed at some length the properties of relative f - normalizers it seems natural also to have at hand the concept of a relative $\mathfrak{F}$ - covering subgroup so that the investigation of the interrelations between the two conjugacy classes in chapter six may be carried out in its fullest generality, that is for a soluble normal subgroup relative to an arbitrary group. Also another advantage of making this characterization our starting point is that it yields a direct method of constructing the $\mathfrak{F}$ - covering subgroups of a given group and relates them to the Sylow systems of that group. If $\mathfrak{F}$ is a local formation and G a group with non-trivial $\mathfrak{F}$-residual R , then a chief factor R/T of G is f-eccentric and complemented; we call a complement of R/T an $\mathfrak{F}$-crucial maximal subgroup. The main theorem of section 5.1 shows that the terminal member of an $\mathfrak{F}$ -crucial maximal chain of G is uniquely determined by a Sylow system $\mathfrak{S}$ of G when we require $\mathfrak{S}$ to reduce into each member of the chain. This unique terminal member (which belongs to $\mathfrak{F}$) we define to be the $\mathfrak{F}$ - covering subgroup of G corresponding to $\mathfrak{S}$. Since we have already shown in chapter four that f - normalizers are the terminal members of another special type of maximal chain, this characterization illustrates certainly an affinity, and in some sense even a duality, between these two canonical classes of formation subgroups. In Theorems 5.2.2 and 5.2.6 we show the $\mathfrak{F}$ - covering subgroups to be a homomorphism-invariant characteristic conjugacy class of G . Theorem 5.2.4 shows an $\mathfrak{F}$ - covering subgroup E of G is also an $\mathfrak{F}$ - covering subgroup of every subgroup that contains it. In section 5.3 we extend the concept of f-abnormality, hitherto applied only to maximal subgroups, to arbitrary subgroups, and in Theorem 5.3.3 show that $\mathfrak{F}$ - covering subgroups are precisely the f-abnormal $\mathfrak{F}$-subgroups of a group. In Theorems 5.3.1 and 5.3.4 we prove some covering and avoidance properties analogous to those proved for Carter subgroups in [4]. Section 5.4 deals with the p-theory of $\mathfrak{F}$- covering subgroups which turns out to be a special case of what has gone before; it also contains some tentative remarks of an exploratory nature about another canonical conjugacy class of what we call the local $\mathfrak{F}$ - covering subgroups.

The last section of chapter five is aimed at proving the second of the two characterizations mentioned at the outset, namely Theorem 5.5.5 which shows that a subgroup of G is an $\mathfrak{F}$ - covering subgroup if and only if it is a maximal $\mathfrak{F}$-subgroup in every homomorphic image. Thus the $\mathfrak{F}$ - covering subgroups of G represent a measure of how far G and its homomorphic images depart from belonging to the formation $\mathfrak{F}$. It is a consequence of Theorem 4.5.7 that when f is integrated an f - normalizer D of G depends only on the formation $\mathfrak{F}$ and not on the function f which defines it locally, and in this case we call D an $\mathfrak{F}$ - normalizer of G. The $\mathfrak{F}$-normalizers of G also provide a yardstick for assessing how far G departs from $\mathfrak{F}$; for a group belongs to $\mathfrak{F}$ if and only if all its chief factors are f-central and we show in chapter four that an $\mathfrak{F}$ - normalizer D of G picks out the f-central chief factors in a given chief series of G and discards the rest. In fact G induces the same group of automorphisms on an f-central chief factor H/K of G as D induces on $H \cap D / K \cap D$ and so figuratively speaking an $\mathfrak{F}$ - normalizer could be said to provide a key for dismantling a chief series and reassembling just the f-central chief factors unaltered.

In chapter six we are interested in connections between $\mathfrak{F}$ - covering subgroups and $\mathfrak{F}$ - normalizers. Theorem 6.1.5 shows that an $\mathfrak{F}$ - normalizer D is contained in the $\mathfrak{F}$ - covering subgroup E corresponding to the same Sylow system and Theorem 6.2.5 shows that D is f-subnormal in E. Abnormality is not a sufficient condition for an $\mathfrak{F}$ - normalizer to be an $\mathfrak{F}$ - covering subgroup (Example 6.2.1) but Theorems 6.2.2 and 6.2.6 do offer necessary and sufficient conditions for this to happen. If M is a maximal subgroup of G supplementing $F(G)$ then an $\mathfrak{F}$ - covering subgroup of M has the form $M \cap E$ for a suitable $\mathfrak{F}$ - covering subgroup E of G (Theorem 6.2.3), and in consequence if X is either an $\mathfrak{F}^*$-normalizer of G or a Hall ϖ-complement when G has a normal Hall ϖ-subgroup, $X \cap E$ is an $\mathfrak{F}$ - covering subgroup of X for suitable choice of E, (Theorems 6.3.1 and 6.3.2). If $G \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$ an $\mathfrak{F}$-normalizer D of G is contained in a unique $\mathfrak{F}$ - covering subgroup E which is the f-subnormalizer of D in G; two $\mathfrak{F}$-normalizers of G contained in E are conjugate in E; and any minimal member of an f-abnormal maximal chain of G is sandwiched between an $\mathfrak{F}$ - normalizer and an $\mathfrak{F}$ - covering subgroup. If $G \in \mathfrak{N}\mathfrak{F}$ the covering and avoidance property characterizes $\mathfrak{F}$ - normalizers but this is not so in general.

Theorem 7.2.8, the main result of chapter seven, gives information about special kinds of subgroup which preserve certain invariants of a finite soluble group. One of these invariants is the nilpotent ϖ-length whose basic properties are described in detail in the first section of chapter seven. One of the several consequences of the main theorem which are discussed in the final section is that a finite soluble group has a pair of $\mathfrak{F}$ - covering subgroups which together generate a subgroup with the same nilpotent ϖ-length and the same set of primes dividing its order as G.

We make frequent, and often tacit, appeal to the Jordan-Hölder theorem in its most general operator form, to the standard isomorphism theorems, and also to the well-known Dedekind relation that if A permutes as a subgroup with B and is contained in C then $A(B \cap C) = AB \cap C$.

Chapter Two

PRELIMINARY RESULTS AND DEFINITIONS

2.1 Notation and Terminology. Groups and complexes are usually denoted by capital Roman letters and their elements by small Roman letters. $|X|$ denotes the cardinal of X, and if this is finite $\sigma(X)$ denotes the set of distinct primes dividing $|X|$. We take $a^b = b^{-1}ab$, $[a,b] = a^{-1}a^b$, and for complexes X and Y of a group we define $[X,Y] = \langle [x,y] \mid x \in X, y \in Y \rangle$, where $\langle g \mid \ldots \rangle$ denotes the group generated by the elements g to be specified. We use braces $\{ \}$ to denote sets and define

$$N_Y(X) = \{ y \mid y \in Y, \text{ and } x^y \in X \text{ for all } x \in X \}, \text{ and}$$
$$C_Y(X) = \{ y \mid y \in Y, \text{ and } x^y = x \text{ for all } x \in X \}.$$

If ϖ is a set of primes, ϖ' is the complementary set, and X is a ϖ-group if $\sigma(X) \leq \varpi$. We frequently refer to the following subgroups of a group: $G' = [G,G]$ is the derived group of G; $F(G)$ is the Fitting subroup of G, that is the largest nilpotent normal subgroup of G; $Z(G)$ is the centre of G; $\phi(G)$ is the Frattini subgroup of G, that is the intersection of all the maximal subgroups of G; $O_\varpi(G)$ is the largest normal ϖ-subgroup of G, and $O_{\underline{\varpi}}(G) = F(G) \cap O_\varpi(G)$ is the largest nilpotent normal ϖ-subgroup of G; $O_{\varpi'\underline{\varpi}}(G)$ is defined by $O_{\varpi'\underline{\varpi}}(G)/O_{\varpi'}(G) = O_{\underline{\varpi}}(G/O_{\varpi'}(G))$; 1 is used to denote the identity subgroup as well as the numeral 'one'. We use the following symbols to denote relations between subsets and subgroups: $X \leq Y$ means X is a subset (subgroup) of Y and $X < Y$ means inclusion is strict; also (a) $X \lessdot Y$, (b) $X \rtimes Y$, (c) $X \lhd Y$ and (d) $X \mathrel{\lhd\!\lhd} Y$ mean in turn that is (a) a maximal, (b) an abnormal, (c) a normal and (d) a characteristic subgroup of Y. An oblique line through these symbols denotes negation. If X and Y are subgroups of a group, $X \perp Y$ means $XY = YX$, that is X permutes with Y. $X \times Y$ denotes the direct product and $X \wr Y$ the wreath product of X and Y (where the permutation representation of Y is to be specified). If $X \leq Y$, the core of X in Y (written $\mathrm{Core}_Y(X)$) is the intersection of all the conjugates of X in Y, and is therefore the largest normal subgroup of Y contained in X. Σ_n denotes the symmetric group of degree n and A_n the corresponding alternating group; C_n denotes the cyclic group of order n and $GL(n,q)$ the group of non-singular n-square matrices with entries in the field $GF(q)$ of q elements. If $L \triangleleft G$ we say X supplements L in G if $LX = G$, and complements L in G if in addition $L \cap X = 1$. A Hall ϖ-subgroup of a group is a ϖ-subgroup whose order is prime to its index, and a Hall ϖ-complement is a Hall ϖ'-subgroup. If ϖ is a single prime p we use the terms Sylow p-subgroup and Sylow p-complement. If $K \triangleleft G$, $K \leq H \triangleleft G$ we call H/K a factor of G and a chief factor when H/K is a minimal normal subgroup of G/K. $\mathrm{Aut}_X(H/K)$ denotes the set of automorphisms induced on H/K through inner automorphisms of G by elements of the set $X \leq G$. $\mathrm{Aut}(G)$ denotes the full group of automorphisms of G. Frequent use is made of the concepts of a subgroup 'covering' and 'avoiding' a factor, and for a detailed account of these ideas we refer to Taunt, [19], p.25. All groups considered are finite.

<u>Classes and Closure Operations</u>. We shall frequently take advantage of this notation introduced by P. Hall (see for example [14], p.533). A class

of groups $\mathfrak{X}$ is a set of isomorphism classes containing groups of order 1. $\mathfrak{X}\mathfrak{Y}$ will denote the class of groups G such that $K \triangleleft G$ with $K \in \mathfrak{X}$ and $G/K \in \mathfrak{Y}$. We write $\mathfrak{X}^2$ for $\mathfrak{X}\mathfrak{X}$. A closure operation C maps classes of groups to classes of groups and satisfies (i) $C(C\mathfrak{X}) = C\mathfrak{X}$ and (ii) $C\mathfrak{X} \leq C\mathfrak{Y}$ whenever $\mathfrak{X} \leq \mathfrak{Y}$ for all classes $\mathfrak{X}$ and $\mathfrak{Y}$. We shall frequently use the closure operations S, Q, R_0, N_0 defined by

$G \in S\mathfrak{X} \iff$ G is isomorphic with a subgroup of an $\mathfrak{X}$-group;

$G \in Q\mathfrak{X} \iff$ G is a homomorphic image of an $\mathfrak{X}$-group;

$G \in R_0\mathfrak{X} \iff$ G has normal subgroups $N_1, \ldots, N_k$ such that $G/N_i \in \mathfrak{X}$, $i = 1, 2, \ldots, k$, and $\bigcap_{i=1}^{k} N_i = 1$;

$G \in N_0\mathfrak{X} \iff$ G has subnormal $\mathfrak{X}$-subgroups $K_1, \ldots, K_r$ such that $\langle K_1, \ldots, K_r \rangle = G$. (We should like to emphasise that this definition of N_0-closure applies only to classes of finite groups.)

If $\mathfrak{X} = C\mathfrak{X}$ for some closure operation C we say that $\mathfrak{X}$ is C-closed. For any group G we use the notation $C(G)$ to mean $C\mathfrak{X}$ where $\mathfrak{X}$ is the class containing G and 1. If $A, B, \ldots, C$ are closure operations, then $\{A, B, \ldots, C\}\mathfrak{X}$ denotes the intersection of classes $\mathfrak{Y}$ such that $\mathfrak{Y} \geq \mathfrak{X}$ and $\mathfrak{Y} = A\mathfrak{Y} = B\mathfrak{Y} = \ldots = C\mathfrak{Y}$.

For easy reference we assign fixed symbols to certain well-known classes of groups which occur in the sequel, and mention a few of their well-known closure properties:

$\mathfrak{A}$ denotes the class of Abelian groups; $\mathfrak{A} = \{S, Q, R_0\}\mathfrak{A}$, but is not N_0-closed;

$\mathfrak{R}_{\varpi}$ denotes the class of ϖ-groups; $\mathfrak{R}_{\varpi} = \{S, Q, R_0, N_0\}\mathfrak{R}_{\varpi}$;

$\mathfrak{N}^p$ denotes the class of p-nilpotent groups, that is the class of groups with a normal Sylow p-complement; $\mathfrak{N}^p = \{ S, Q, R_0, N_0 \} \mathfrak{N}^p$;

$\mathfrak{N}$ denotes the class of nilpotent groups; $\mathfrak{N} = \{ S, Q, R_0, N_0 \} \mathfrak{N}$;

$\mathfrak{U}$ denotes the class of supersoluble groups; $\mathfrak{U} = \{ S, Q, R_0 \} \mathfrak{U}$;

it is well-known that $\mathfrak{U}$ is not N_0-closed;

$\mathfrak{S}$ denotes the class of soluble groups; $\mathfrak{S} = \{ S, Q, R_0, N_0 \} \mathfrak{S}$.

We have the following relations between these classes:

$$\mathfrak{A} \cup \mathfrak{R}_p < \mathfrak{N} < \mathfrak{U} < \mathfrak{N}\mathfrak{A} < \mathfrak{S} = \mathfrak{S}^2 .$$

2.2 We now discuss the elements of the theory of formations introduced and developed by Gaschütz in [8].

2.2.1 DEFINITIONS. A class of groups $\mathfrak{X}$ is called a formation if $\mathfrak{X} \leq \mathfrak{S}$ and $\mathfrak{X} = \{ Q, R_0 \} \mathfrak{X}$; here we differ slightly from Gaschütz by excluding the possibility of empty formations. If, in addition, $G \in \mathfrak{X}$ whenever $G/\Phi(G) \in \mathfrak{X}$, $\mathfrak{X}$ is called a saturated formation. The equivalence of this definition to that given in [8] was proved by Gaschütz and Lubeseder in [9]. If $f(p)$ is a specified formation for each prime p, f is called a formation function; in other words, a formation function is a mapping from the set of all primes into the set of formations. Corresponding to a given formation $f(p)$ we define a class $\mathfrak{F}_p$ of soluble groups by

$$G \in \mathfrak{F}_p \iff G/O_{p'p}(G) \in f(p).$$

$\mathfrak{F}_p$ is called the p-local formation defined by $f(p)$. The class $\mathfrak{F} = \bigcap_p \mathfrak{F}_p$ is called a local formation, or, more precisely, the formation defined locally by f ; for $\mathfrak{F}_p$ and $\mathfrak{F}$ are certainly formations, and Gaschütz has shown ([8], Satz 3.1) that they are saturated. Moreover, Gaschütz and Lubeseder have proved in as yet unpublished work that conversely every saturated formation may be locally defined, (a proof of this result may be found in chapter 4 of [18]). Hence the terms 'saturated' and 'local' may be used synonymously when qualifying formations. If C is a closure operation and $f(p) = Cf(p)$ for all primes p, we say f is a <u>C-closed formation function</u>, and use the notation $f = Cf$.

<u>2.2.2</u> <u>LEMMA</u>. If $f = Sf$, then $\mathfrak{F}_p = S\mathfrak{F}_p$ and $\mathfrak{F} = S\mathfrak{F}$.

<u>Proof</u>. Let X be a subgroup of $G \in \mathfrak{F}_p$, and write $K = O_{p'p}(G)$, the product of all the normal $\mathfrak{N}^p$-subgroups of G. Now $K \cap X$ is a normal $\mathfrak{N}^p$-subgroup of X, and therefore $K \cap X \leq O_{p'p}(X)$. Hence by the isomorphism theorem we have $X/O_{p'p}(X) \in Q(X/K \cap X) = Q(KX/K) \leq QS(G/K) \leq QSf(p) = f(p)$ by hypothesis. Hence $X \in \mathfrak{F}_p = S\mathfrak{F}_p$. The last statement follows from the fact that an arbitrary collection of S-closed classes has S-closed intersection.

<u>2.2.3</u> <u>DEFINITION</u>. If $\mathfrak{X}$ is a formation, the intersection of all normal subgroups K of G with $G/K \in \mathfrak{X}$ is called the <u>$\mathfrak{X}$-residual of G</u> and is written $R_{\mathfrak{X}}(G)$. Clearly we have $R_{\mathfrak{X}}(G) \trianglelefteq G$.

If $\mathfrak{F}$ is a formation defined locally by f, we have the following simple consequence of the above definitions.

<u>2.2.4</u> <u>LEMMA</u>. $G \in \mathfrak{F} \iff R_{f(p)}(G) \in \mathfrak{N}^p$ for all primes p.

Although our next result is elementary, it plays an important part in the sequel.

2.2.5 LEMMA. If $f^*(p) = \mathfrak{F} \cap f(p)$ for all primes p, then the formation function f^* also defines $\mathfrak{F}$ locally.

Proof. Let $\mathfrak{F}^*$ be the local formation defined by f^*. It is clear that $\mathfrak{F}^* \leq \mathfrak{F}$. To show the reverse inclusion let $G \in \mathfrak{F}$; then for every p we have $G/O_{p'p}(G) \in f(p)$. But $G/O_{p'p}(G) \in Q\mathfrak{F} = \mathfrak{F}$, and therefore $G/O_{p'p}(G) \in \mathfrak{F} \cap f(p) = f^*(p)$. Hence $G \in \mathfrak{F}^*$ and $\mathfrak{F} \leq \mathfrak{F}^*$ as claimed.

2.2.6 DEFINITION. If $f(p) \leq \mathfrak{F}$ for each prime p, we call $\{f(p)\}$ a set of integrated formations and f an integrated formation function. Lemma 2.2.5 shows that every local formation may be defined by an integrated formation function.

We end this section by mentioning two well-known saturated formations which will serve as illustrations in later work.

2.2.7 LEMMA. (a) If $f(p) = 1$ for all p, then f defines $\mathfrak{N}$ locally. (b) If $f(p)$ = the class of $\mathfrak{A}$-groups of exponent p-1 for all p, then f defines $\mathfrak{U}$ locally.

Proof. (a) If $G \in \mathfrak{F}$, since $f(p) = 1$ we have $G = R_{f(p)}(G) \in \mathfrak{N}^p$ by (2.2.4) for all primes p. Hence $\mathfrak{F} \leq \bigcap_p \mathfrak{N}^p = \mathfrak{N}$. But evidently we have $\mathfrak{N} \leq \mathfrak{F}$ and the result follows.

(b) In [2], p.183 Theorem 1, Baer shows that a group is supersoluble if and only if for every chief factor H/K of G $\mathrm{Aut}_G(H/K)$ is an $\mathfrak{A}$-group of exponent p-1. Since $O_{p'p}(G)$ is well-known to be the centralizer of the p-chief factors of a soluble group G (see for example Huppert, [17], p.513 Hilfssatz 6), Baer's criterion is equivalent to the condition that

$G/O_{p'p}(G)$ shall be Abelian of exponent p-1, for this condition certainly implies that $G \in \mathfrak{N}\mathfrak{A} < \mathfrak{S}$. This completes the proof.

2.3 From this point on until the end of chapter six G will denote an arbitrary finite group, and L a soluble normal subgroup of G .

2.3.1 DEFINITIONS. If M is a maximal subgroup of G with $|G:M|$ a power of some prime p, we say M is p-maximal in G . If H/K is a factor of G such that $K \leq M$ and $HM = K$, we say M supplements H/K ; if, in addition, $H \cap M = K$, we say M complements H/K . Accordingly we call H/K a supplemented or complemented factor.

Although our next three results are thinly disguised forms of well-known propositions, for completeness we sketch their proofs.

2.3.2 LEMMA. Let N be a soluble minimal normal subgroup of G supplemented by a maximal subgroup M of G . Then M complements N , M is p-maximal in G , and if K is a normal subgroup of G we have $C_K(N) \cap M = K \cap \mathrm{Core}_G(M)$. If N^* is another minimal normal subgroup of G supplemented by M , then $N^* \underset{K}{\cong} N$.

Proof. It is well-known that N is an elementary Abelian p-group for some prime p; hence $M \cap N$ is centralized by N , is normalized by M and is therefore normal in $MN = G$. Thus by minimality $M \cap N = 1$, and so $|G:M| = |N|$, a power of p. $C_K(N) \cap M$ is centralized by N , normalized by M and is therefore normal in $MN = G$. Hence $C_K(N) \cap M \leq K \cap \mathrm{Core}(M)$. Since $N \nleq M$, we have $[N, K \cap \mathrm{Core}(M)] \leq N \cap K \cap \mathrm{Core}(M) = 1$, and therefore $K \cap \mathrm{Core}(M) \leq C_K(N) \cap M$. This proves $C_K(N) \cap M =$

$K \cap \mathrm{Core}(M)$ as claimed. To prove the last part of the statement, first suppose $M \cap N^* = T \neq 1$. Then $T \triangleleft M$. But $[N,N^*] \leq N \cap N^* = 1$, and therefore $T \triangleleft MN = G$. Hence by minimality of N^* we have $T = N^*$ which contradicts the assumption that $N^* \nleq M$. Thus $T = 1$, and $|N^*| = |G:M|$ is a power of p; therefore since N^* is a p-group it is soluble, and by the above argument $C_K(N^*) \cap M = K \cap \mathrm{Core}(M)$. With $K = G$ it follows that $N\,\mathrm{Core}(M) = N^*\,\mathrm{Core}(M)$, $= X$ say. Each coset of $\mathrm{Core}(M)$ in X contains a unique element n of N and a unique element n^* of N^* and the correspondence $n \longleftrightarrow n^*$ gives a G-operator-isomorphism between N and N^*.

2.3.3 LEMMA. Let M be a maximal subgroup of G such that $ML = G$. Then M is p-maximal in G for some prime p and G has a chief factor U/V which is complemented by M. If H/K is any chief factor of G supplemented by M, then it is complemented by M, and we have $\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_G(U/V)$ and $\mathrm{Aut}_L(H/K) \cong L \cap M \,/\, L \cap \mathrm{Core}(M)$.

Proof. Let $1 = G_0 < G_1 < \ldots < G_r = L$ be part of a chief series of G. Since $L \nleq M$, there exists an integer i, $0 \leq i \leq r$, such that $G_i \leq M$ and $G_{i+1} \nleq M$. Write $U = G_{i+1}$ and $V = G_i$. Then U/V is a soluble chief factor of G supplemented by M. We now apply (2.3.2) to U/V and conclude that M complements U/V and that $|G:M| = |U:V| = p^{\alpha}$ for some prime p; this proves the first statement. Now put $\bar{U} = KU$ and $\bar{V} = KV$. Then $U \cap \bar{V}$ is a normal subgroup of G containing V and contained in U; therefore $U \cap \bar{V}$ is either U or V. Since $U \cap \bar{V} \leq M$ and $U \nleq M$, we must have $U \cap \bar{V} = V$. We can now apply the operator form of the isomorphism theorem with the automorphisms induced by the elements of G as the group of operators, and deduce that $\bar{U}/\bar{V} \underset{G}{\cong} U/U \cap KV = U/V$. By a similar argument we have $H/K = H/H \cap KV \underset{G}{\cong} HV/\bar{V}$. If we now apply (2.3.2) to $G/\bar{V}$ with $\bar{U}/\bar{V}$ in the rôle of N and $HV/\bar{V}$ in the rôle of N^* , it follows that $Aut_G(H/K) \cong Aut_G(U/V)$. Finally, since we now have $Aut_L(H/K) \cong Aut_L(U/V)$, we show $Aut_L(U/V) \cong L \cap M/L \cap Core(M)$. We have $(L \cap M)U = L \cap MU = L \cap G = L$, and therefore since $U \leq C_L(U/V) \leq L$ we have $(L \cap M)C_L(U/V) = L$. Hence $Aut_L(U/V) \cong L/C_L(U/V) \cong L \cap M/C_L(U/V) \cap M$. By (2.3.2) with $K = L$ we have $C_L(U/V) \cap M = L \cap Core(M)$. This completes the proof.

It is convenient to label the following well-known result.

<u>2.3.4</u> <u>LEMMA</u>. Let H/K be a chief factor of G such that $H/K \nleq \emptyset(G/K)$; then H/K is supplemented in G .

<u>2.3.5</u> <u>DEFINITIONS</u>. (a) We recall the usual terminology that H/K is a <u>p-chief factor</u> when $|H:K|$ is a power of the prime p. We say H/K is <u>below</u> L if $H \leq L$ and <u>above</u> L if $K \geq L$. We call an arbitrary chief factor H/K of G <u>f-central for L</u> if either

(i) L avoids H/K , or

(ii) L covers H/K and $Aut_L(H/K) \in f(p)$ where $p \mid |H:K|$.

Otherwise we say H/K is <u>f-eccentric for L</u> .

<u>Remarks</u>. We note that if L avoids H/K, then $[L,H] \leq L \cap H \leq K$ and therefore $Aut_L(H/K) = 1$. If, on the other hand, L covers H/K then H/K is contained in the soluble group LK/K , and therefore by (2.3.2) H/K is a p-chief factor for some prime p. Thus (ii) is a well-

defined contingency. When $f(p) = 1$ for all primes p, the terms f-central and f-eccentric coincide with the concepts central and eccentric introduced by P. Hall in section 5 of [13].

(b) We call a maximal subgroup M of G <u>f-normal for L</u> if either

(i) $L \leq M$, or

(ii) $LM = G$ and $L \cap M/L \cap \mathrm{Core}(M) \in f(p)$ where $p \mid |G:M|$,

and <u>f-abnormal for L</u> otherwise.

<u>Remarks</u>. By (2.3.3) M is p-maximal for some prime p in case (ii), and therefore this alternative is well-defined. If we take $L = G$, omitting the qualification 'for L' from the terminology, and if $f(p) = 1$ for all primes p, it is clear that the terms f-normal and f-abnormal reduce to the usual concepts normal and abnormal as applied to maximal subgroups - for a non-normal maximal subgroup is the same as an abnormal maximal subgroup.

The next theorem shows a close connection between the concept 'f-normal for L' for a maximal subgroup and the concept 'f-central for L' for a chief factor.

<u>2.3.6</u> <u>THEOREM</u>. If M is a maximal subgroup of G f-normal for L and if M supplements a chief factor H/K of G, then H/K is f-central for L. Conversely, if H/K is a chief factor of G which is f-central for L and supplemented by a maximal subgroup M of G, then M is f-normal for L.

<u>Proof</u>. Let M be a maximal subgroup f-normal for L supplementing H/K. If $L \leq M$ then $LK \leq M$, and since $H \nleq M$ we have $H \cap LK = K$.

In this case $[H,L] \leq H \cap LK = K$ and $L = C_L(H/K)$. Therefore $Aut_L(H/K) = 1$, and H/K is certainly f-central for L . Otherwise $LM = G$ and by (2.3.3) M is p-maximal and H/K is a p-chief factor, for some prime p; also by (2.3.3) we have $Aut_L(H/K) \cong L \cap M/L \cap Core(M) \in f(p)$, and therefore H/K is f-central for L . Conversely, let H/K be a chief factor of G f-central for L complemented by M . If $LM = G$ we have $L \cap M/L \cap Core(M) \cong Aut_L(H/K) \in f(p)$ where $p \mid |H:K| = |G:M|$ and therefore M is f-normal for L .

It follows at once from (2.3.6) that a maximal subgroup is f-abnormal for L if and only if it complements a chief factor of G f-eccentric for L .

<u>2.3.7</u> <u>LEMMA</u>. Let M be a maximal subgroup of G supplementing L , and write $W = L \cap Core(M)$. Then L/W contains exactly one minimal normal subgroup V/W of G/W . Moreover, the chief factor V/W is complemented in G by M and is self-centralizing in L ; in particular, we have $L/V \cong L \cap M/W$.

<u>Proof</u>. Without loss of generality set $W = 1$. Since $LM = G$, L is certainly non-trivial and therefore contains a minimal normal subgroup V of G . Since $L \cap Core\,(M) = 1$, M contains no normal subgroup of G contained in L , and therefore $MV = G$. Hence by (2.3.2) $M \cap V = 1$ and $|G:M| = |V| = p^{\alpha}$ for some prime p. Since $C_L(V) \cap M = L \cap Core\,(M) = 1$, by (2.3.2) we have $C_L(V) = C_L(V) \cap MV = V(C_L(V) \cap M) = V$. If V^* is a minimal normal subgroup of G contained in L , we

have $V^* \leq C_L(V) = V$ and therefore $V^* = V$. This completes the proof.

We end this section with another well-known result which turns out to be very useful in the sequel.

2.3.8 LEMMA. If M is a maximal subgroup of G which supplements a nilpotent normal subgroup K of G, then $K/K \cap M$ is a chief factor of G.

Proof. Since $KM = G$ we have $K \cap M \neq K$. By a well-known property of nilpotent groups $K \cap M < N_K(K \cap M)$, and therefore $K \cap M \lhd \langle M, N_K(K \cap M) \rangle = G$. If G had a normal subgroup, H say, strictly between K and $K \cap M$ we should have $M < HM < G$ contradicting the maximality of M. Hence $K/K \cap M$ is a chief factor of G.

2.4 Most of the results in this section are straightforward extensions of well-known properties of soluble groups.

2.4.1 DEFINITION. We denote the intersection of all the p-maximal subgroups of G by $\phi_p(G)$ and call it the p-Frattini subgroup of G. Since every maximal subgroup of a soluble group is p-maximal for some prime p, it is clear that if G is soluble $\phi(G) = \bigcap_p \phi_p(G)$. However, we are here concerned with the situation where G is an arbitrary finite group and L a soluble normal subgroup.

2.4.2 LEMMA. (a) $\phi_p(G) \cap L$ is a normal $\mathfrak{N}^p$-subgroup of G.
(b) $(\bigcap_p \phi_p(G)) \cap L = \phi(G) \cap L \geq \phi(L)$.

Proof. (a) Write $J = \Phi_p(G) \cap L$. If M is a p-maximal subgroup of G so is M^α for $\alpha \in Aut(G)$, and therefore $\Phi_p(G) \trianglelefteq G$; hence $J \triangleleft G$. Since J is soluble, by Theorem 1 of [10] it has a Sylow p-complement, S say, and all such are conjugate in J. Write $N = N_G(S)$ and let $g \in G$. Since $J \triangleleft G$, S^g is also a Sylow p-complement of J. Therefore $S^g = S^x$ with $x \in J$, and hence $gx^{-1} \in N$. Hence $g \in NJ$ and since g was arbitrary $G = NJ$. (This well-known method of proof will be referred to as the 'Frattini argument' in the sequel.) Suppose $N \neq G$. We have $|G:N| = |NJ:N| = |J:J \cap N|$ which is a power of p since $S \leq J \cap N$. Hence N is contained in a p-maximal subgroup M of G such that $MJ = G$. But $J \leq \Phi_p(G) \leq M$, and we have a contradiction. It therefore follows that $N = G$, $S \triangleleft G$ and that J is p-nilpotent as claimed.

(b) Certainly $\Phi(G) \leq \Phi_p(G)$ for each p, and therefore writing $K = (\bigcap_p \Phi_p(G)) \cap L$ we have $\Phi(G) \cap L \leq K$. To prove the reverse inequality, and hence equality, it is enough to show that K is contained in every maximal subgroup of G. Let $M \lessdot G$. If $L \leq M$, a fortiori $K \leq M$. On the other hand, if M supplements L, by (2.3.3) M is p-maximal for some prime p, and therefore contains K as desired. The statement $\Phi(L) \leq \Phi(G) \cap L$ follows from Satz 5 of [7].

The next lemma depends entirely on results of Gaschütz in [7].

2.4.3 LEMMA. Let $L^* = \Phi(G) \cap L$. Then $F(L)/L^*$ is the Fitting subgroup of L/L^*, and is the intersection of L/L^* with the socle of G/G^*.

<u>Proof</u>. The assertion $F(L)/L^* = F(L/L^*)$ follows directly from Satz 10 of [7]. To prove the second part of the lemma we may write $L^* = 1$ without loss of generality; for by Satz 2 of [7], $\phi(G/L^*) = \phi(G)/L^*$. By Satz 5 of [7], $\phi(F(L)) \leq \phi(G) \cap L = 1$, and therefore $F(L)$ is a normal $\mathfrak{A}$-subgroup of G intersecting $\phi(G)$ trivially. Hence by Satz 7 of [7], $F(L)$ is completely reducible qua G-module, and is therefore the direct product of minimal normal $\mathfrak{A}$-subgroups of G. The second part of the statement now follows easily.

For the rest of this chapter we use J (and occasionally J_p) to denote the subgroup $\phi_p(G) \cap L$.

<u>2.4.4</u> <u>LEMMA</u>. The Fitting subgroup of $\bar{L} = L/J$ is $O_{p'p}(L)/J$, and is the direct product of minimal normal p-subgroups of G/J.

<u>Proof</u>. Write $T = O_{p'}(L)$. Then we have the relation $T \leq J \leq O_{p'p}(L)$; for the left-hand inclusion follows from the fact that every p-maximal subgroup of G contains T, and the right-hand inclusion from the fact that $J \in \mathfrak{N}^p = N_0\mathfrak{N}^p$. Since L/T has no normal p'-subgroups, it is clear that its Fitting subgroup is a p-group, and therefore $F(L/T) = O_{p'p}(L)/T$. Moreover since every maximal subgroup of G/T contains J/T, we have $\phi(G/T) \cap L/T = J/T$. Therefore by (2.4.3) $F(\bar{L}) = O_{p'p}(L)/J$ which proves the first assertion; the second is immediate from (2.4.3).

Recalling that $\mathfrak{F}_p$ denotes the p-local formation defined by the formation $f(p)$ we have

Proof. The assertion $F(L)/L^* = F(L/L^*)$ follows directly from Satz 10 of [7]. To prove the second part of the lemma we may write $L^* = 1$ without loss of generality; for by Satz 2 of [7], $\phi(G/L^*) = \phi(G)/L^*$. By Satz 5 of [7], $\phi(F(L)) \leq \phi(G) \cap L = 1$, and therefore $F(L)$ is a normal $\mathfrak{A}$-subgroup of G intersecting $\phi(G)$ trivially. Hence by Satz 7 of [7], $F(L)$ is completely reducible qua G-module, and is therefore the direct product of minimal normal $\mathfrak{A}$-subgroups of G. The second part of the statement now follows easily.

For the rest of this chapter we use J (and occasionally J_p) to denote the subgroup $\phi_p(G) \cap L$.

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Proof. Write $T = O_{p'}(L)$. Then we have the relation $T \leq J \leq O_{p'p}(L)$; for the left-hand inclusion follows from the fact that every p-maximal subgroup of G contains T, and the right-hand inclusion from the fact that $J \in \mathfrak{N}^p = N_o\mathfrak{N}^p$. Since L/T has no normal p'-subgroups, it is clear that its Fitting subgroup is a p-group, and therefore $F(L/T) = O_{p'p}(L)/T$. Moreover since every maximal subgroup of G/T contains J/T, we have $\phi(G/T) \cap L/T = J/T$. Therefore by (2.4.3) $F(\bar{L}) = O_{p'p}(L)/J$ which proves the first assertion; the second is immediate from (2.4.3).

Recalling that $\mathfrak{F}_p$ denotes the p-local formation defined by the formation $f(p)$ we have

2.4.5 THEOREM. $L \in \mathfrak{F}_p$ if and only if every minimal normal subgroup of $\bar{G} = G/J$ contained in $\bar{L} = L/J$ is f-central for $\bar{L}$ (or, equivalently, f-central for L when considered as a chief factor of G).

Proof. We first prove the sufficiency of the condition. If $J = L$ then by (2.4.2 (a)) $L \in \mathfrak{N}^p$, and since $1 \in f(p)$ we have $L \in \mathfrak{F}_p$. We therefore suppose that $J \neq L$ so that by (2.4.4) we may write

$$F(\bar{L}) = \bar{N}_1 \times \bar{N}_2 \times \dots \times \bar{N}_r ,$$

where $\bar{N}_i = N_i/J$ are minimal normal subgroups of $\bar{G}$ contained in $\bar{L}$. It is a well-known property of a finite soluble group that the Fitting subgroup contains its centralizer. Thus, writing $\bar{C}_i = C_{\bar{L}}(\bar{N}_i)$, we have $\bar{C} = \bigcap_{i=1}^{r} \bar{C}_i \leq F(\bar{L})$. Since $F(\bar{L}) \in \mathfrak{A}$, we therefore obtain $\bar{C} = F(\bar{L})$. Hence if each $\bar{N}_i$ is f-central for $\bar{L}$, we have $\bar{L}/\bar{C}_i \in f(p)$, $i = 1, 2, \dots, r$, and therefore $\bar{L}/\bar{C} \in R_0 f(p) = f(p)$. Hence $L/O_{p'p}(L) \cong (L/J) / (O_{p'p}(L)/J) = \bar{L}/\bar{C} \in f(p)$, and therefore $L \in \mathfrak{F}_p$. To prove the necessity let $L \in \mathfrak{F}_p$, and without loss of generality assume $J \neq L$. Then $L/O_{p'p}(L) \in f(p)$, and since $F(\bar{L}) \leq \bar{C}_i$, we have $\bar{L}/\bar{C}_i \in Q(\bar{L}/F(\bar{L})) = Q(L/O_{p'p}(L)) \leq Qf(p) = f(p)$. Therefore $\bar{N}_i$ is f-central for $\bar{L}$, $i = 1, 2, \dots, r$, and the proof is complete

Remark. We note in passing that we have just shown $O_{p'p}(L)$ to be the intersection of L and the centralizers in G of those chief factors of G lying between J and $O_{p'p}(L)$. Therefore, as all the $\bar{N}_i$ are complemented chief factors of G by (2.3.4), $O_{p'p}(L)$ may be considered as the intersection of the centralizers in L of just the supplemented chief factors of G . This generalizes in one direction Hilfssatz 6 of [17]. It then follows that $F(L)$ is the intersection of the centralizers in L of the complemented chief factors of G.

2.4.6 THEOREM. The following conditions are equivalent:

(i) $L \in \mathfrak{F}_p$;

(ii) Every p-chief factor of G is f-central for L ;

(iii) Every p-maximal subgroup of G is f-normal for L ;

(iv) $L/J_p \in \mathfrak{F}_p$.

Proof. (i) ⇒ (ii). If $L \in \mathfrak{F}_p$, then $L/O_{p'p}(L) \in f(p)$, and since $O_{p'p}(L)$ centralizes every p-chief factor H/K of G, we have $\mathrm{Aut}_L(H/K) \in Qf(p) = f(p)$ as required. (ii) ⇒ (iii). This follows from (2.3.6). (iii) ⇒ (i). Since by (2.3.4) each chief factor of G between J_p and $O_{p'p}(L)$ is complemented by a p-maximal subgroup which by hypothesis is f-normal for L, by (2.3.6) each is f-central for L, and therefore by (2.4.5) $L \in \mathfrak{F}_p$. Hence (i), (ii) and (iii) are equivalent. (i) ⇒ (iv). This follows from the fact that $\mathfrak{F}_p = Q\,\mathfrak{F}_p$. (iv) ⇒ (i). From (i) ⇒ (ii) it follows that if $L/J_p \in \mathfrak{F}_p$ then every chief factor of G between J_p and $O_{p'p}(L)$ is f-central for L, and therefore by (2.4.5) $L \in \mathfrak{F}_p$.

2.4.7 THEOREM. The following statements are equivalent:

(i) $L \in \mathfrak{F}$;

(ii) Every chief factor of G is f-central for L ;

(iii) Every maximal subgroup of G is f-normal for L ;

(iv) $L/\emptyset(G) \cap L \in \mathfrak{F}$.

<u>Proof</u>. The equivalence of the first three statements follows easily from (2.4.6) by letting p run through $\sigma(L)$ and recalling the definition $\mathfrak{F} = \bigcap_p \mathfrak{F}_p$. <u>(i) $\Rightarrow$ (iv)</u>. This follows from the fact that $\mathfrak{F} = Q\mathfrak{F}$. <u>(iv) $\Rightarrow$ (i)</u>. By (2.4.2 (b)) $J_p \geq \Phi(G) \cap L$, and therefore $L/J_p \in Q(L/\Phi(G) \cap L) \leq Q\mathfrak{F} = \mathfrak{F} \leq \mathfrak{F}_p$. Hence $L \in \mathfrak{F}_p$ by (2.4.6), and since this is true for each p we have $L \in \bigcap_p \mathfrak{F}_p = \mathfrak{F}$ as required.

On taking $L = G$ in (2.4.7) we see that a soluble group belongs to $\mathfrak{F}$ if and only if every chief factor is f-central, and if and only if every maximal subgroup is f-normal.

Chapter Three

f(p) - NORMALIZERS

3.1 In this chapter we are concerned with the local or p-theory of f-normalizers, and therefore we shall use the letter p to denote a fixed prime throughout; also, unless otherwise stated, L will denote a soluble normal subgroup of an arbitrary group G as before. In the case $f(p) = 1$ this theory reduces to the study of the normalizers in G of the Sylow p-complements of L, which is not perhaps a very rewarding excercise. However, the greater complexity introduced by taking f(p) to be an arbitrary formation is our justification for pausing to make a more detailed analysis before moving on to the study of f-normalizers in general. Many of the concepts and results of this chapter have analogues in chapter four, and at first glance one might suspect that our local results were just a special case of those proved subsequently about f-normalizers in general. In fact, if we take $\mathfrak{F}$ to be the formation defined locally by f(p) and $f(q) = \mathfrak{S}$ for all $q \neq p$, then $\mathfrak{F}$ coincides with $\mathfrak{F}_p$ and this suspicion becomes stronger. But in order to prove the significant theorems of chapter four which might seem to generalize corresponding results of this chapter, we need the important extra condition that the formation function f defining $\mathfrak{F}$ locally is integrated, and this is clearly not the case when $f(q) = \mathfrak{S}$ for all $q \neq p$ unless $f(p) = \mathfrak{S}$ also. In fact it is not difficult to see that f cannot usually be suitably defined to make the p-theory a special case of the general theory developed in chapter four, and that genuinely additional information is obtained here.

3.1.1 DEFINITION. Let $C_p(L)$ denote the intersection of the centralizers in L of those p-chief factors of G which are f-central for L, and take $C_p(L) = L$ if all the p-chief factors of G are f-eccentric for L. We call $C_p(L)$ the f(p) - centralizer of L (relative to G). It is clear that $C_p(L) \lhd L \lhd G$, and that $L/C_p(L) \in f(p)$. Our next lemma shows that $C_p(L)$ depends only on L and is independent of the group G in which L is embedded. We therefore drop forthwith the parenthetic 'relative to G' from the definition.

3.1.2 LEMMA. $C_p(L)$ is equal to the intersection of the centralizers in L of the f-central p-chief factors of L.

Proof. By the Jordan-Hölder Theorem it is sufficient to consider the chief factors of just one chief series. Let $1 = L_0 < L_1 < \dots < L_r = L < \dots < G$ be a chief series of G through L. Since the p-chief factors above L are f-central for L we see that $C_p(L)$ is by definition the intersection of those $C_L(L_{i+1}/L_i)$ such that $L/C_L(L_{i+1}/L_i) \in f(p)$ and $p \mid |L_{i+1}:L_i|$. That part of the given chief series of G below L may be refined to a chief series of L. Since $L \lhd G$, by Clifford's Theorem, (see for example [6], p.343 Theorem 49.2), a chief factor L_{i+1}/L_i of G decomposes into the direct product $\bar{N}_1 \times \bar{N}_2 \times \dots \times \bar{N}_t$ where $t \geq 1$, and $\bar{N}_j = N_j/L_i$ is a chief factor of L, $1 \leq j \leq t$. Further $\mathrm{Aut}_L(\bar{N}_j) \cong \mathrm{Aut}_L(\bar{N}_k)$, $1 \leq j,k \leq t$, and writing $C = C_L(L_{i+1}/L_i)$ and $C_j = C_L(\bar{N}_j)$, $1 \leq j \leq t$, we have

$$C = \bigcap_{j=1}^{t} C_j . \qquad (*)$$

Therefore, if L_{i+1}/L_i is f-central for L, we have $L/C_j \in Q(L/C) \leq Qf(p) = f(p)$, and hence $\bar{N}_j$, considered as a chief factor of L, is f-central for each j. Conversely, if a single $\bar{N}_j$ is f-central then so are all the $\bar{N}_j$'s, and therefore by condition (*) $L/C \in R_0(L/C_j) \leq R_0 f(p) = f(p)$. Hence we have proved that $\bar{N}_j$ is f-central for each j if and only if L_{i+1}/L_i is f-central for L, and the lemma now follows at once from condition (*).

<u>3.1.3</u> <u>DEFINITIONS</u>. Since L is soluble, it has a Sylow p-complement, L^p say, and all such are conjugate in L. Write $T^p = L^p \cap C_p(L)$. Since $C_p(L) \triangleleft L$, the subgroup T^p is a Sylow p-complement of $C_p(L)$, and all the p-complements of $C_p(L)$ occur in this way. We call T^p the <u>f(p) - complement of L corresponding to L^p</u>, and we observe from (3.1.2) that it is independent of G. Since $C_p(L) \triangleleft L$, the set of all f(p) - complements form a characteristic conjugacy class of L, and therefore a conjugacy class of G. We call $N_G(T^p)$ the <u>f(p) - normalizer of L relative to G corresponding to L^p</u>. If $L = G$, we omit the tag 'relative to G', and in the event of any ambiguity speak of an <u>absolute</u> f(p) - normalizer of L. (Our terminology follows closely that developed by P. Hall in [13].) It is clear from (3.1.2) and the definition that the intersection with L of an f(p) - normalizer of L relative to G is an absolute

$f(p)$ - normalizer of L. Since the T^p form a conjugacy class of G permuted transitively by the inner automorphisms of G induced by the elements of L, the same is true of the $f(p)$ - normalizers of L relative to G. Moreover, if $L \lhd G$ (and in particular if $L = G$), then $C_p(L) \lhd G$, and it is clear that the relative $f(p)$ - normalizers of L form a characteristic conjugacy class of G. We recall that Carter's definition of an abnormal subgroup in [4] is equivalent to the combination of conditions (i) and (ii) of Theorem 3.6 in [13]; that is H is abnormal in G if and only if every subgroup containing H is self-normalizing in G and H is not contained in two distinct conjugates. It therefore follows from Theorem 3.7 of [13] that the relative $f(p)$ - normalizers of L are abnormal subgroups of G. Our next result shows that the subgroup $C_p(L)$ is not uniquely determined by the rôle it has to play, and that other, and in general different, canonical subgroups exist which have $N_G(T^p)$ as the normalizer of one of their Sylow p-complements.

<u>3.1.4</u> <u>LEMMA</u>. If R is a normal subgroup of G contained in $C_p(L)$ such that $L/R \in f(p)$, then $N_G(T^p) = N_G(T^p \cap R)$ where T^p is an $f(p)$ - complement of L.

<u>Proof</u>. Since clearly we have $N_G(T^p) \leq N_G(T^p \cap R)$, it is sufficient to show the two subgroups have the same order. Let H/K be a chief factor of G. Since $T^p \cap R$ is evidently a Sylow p-complement of R, it follows from Theorems 6.2 and 7.2 of [13] that $N_G(T^p \cap R)$

either covers or avoids H/K , and that it fails to cover only when H/K is a p-chief factor not centralized by R . However, if H/K is a p-chief factor centralized by R , it follows from the Q-closure of $f(p)$ that H/K is f-central for L , and is therefore centralized by $C_p(L)$. Hence by a further application of Theorem 7.2 of [13] we see that whenever H/K is covered by $N_G(T^p \cap R)$ it is also covered by $N_G(T^p)$. Thus $|N_G(T^p)| \geq |N_G(T^p \cap R)|$ and the result follows.

Lemma 3.1.4 shows that had we taken $R_{f(p)}(L)$, the $f(p)$ - residual of L , in place of $C_p(L)$, we should have obtained an equivalent definition of a relative $f(p)$ - normalizer; for $R_{f(p)}(L) \trianglelefteq L$ implies $R_{f(p)}(L) \triangleleft G$, and therefore $R_{f(p)}(L)$ can be identified with R in (3.1.4). If $L = G \in f(p)$, then $R_{f(p)}(L) = 1$ and $C_p(L) = O_{p'p}(L)$, and so in general the subgroups T^p and $T^p \cap R$ of (3.1.4) can indeed be quite distinct. It is not difficult to see from the remarks following (2.4.5) that $C_p(L)/R_{f(p)}(L)$ is contained in $O_{p'p}(L/R_{f(p)}(L))$, and is therefore p-nilpotent. This observation, although not relevent to the present line of investigation, can be used to provide the basis of an alternative proof of (3.1.4).

In anticipation of our next result it is perhaps worth pointing out that if M is any p-maximal subgroup of G supplementing L , it is always possible to choose a conjugate of M which contains a prescribed Sylow p-complement of L . The reason is that because $|L:L \cap M|$ is a power of p, $L \cap M$ contains a Sylow p-complement of L , and therefore there is a suitable conjugate $(L \cap M)^g = L \cap M^g$

containing any given Sylow p-complement of L. Our next result shows that every p-maximal subgroup of G f-abnormal for L contains a relative $f(p)$ - normalizer of L.

<u>3.1.5</u> <u>LEMMA</u>. Let $N = N_G(T^p)$ be the $f(p)$ - normalizer of L relative to G corresponding to the Sylow p-complement L^p of L. Suppose $L \notin \mathcal{Y}_p$, and let M be a p-maximal subgroup of G which is f-abnormal for L and contains L^p. Then $L^p \leq N \leq M$.

<u>Proof</u>. L^p normalizes L^p and $C_p(L)$; therefore $L^p \leq N_G(L^p \cap C_p(L)) = N$. To prove the second inclusion write $W = L \cap \mathrm{Core}(M)$. Since by hypothesis $L \leq M$, by (2.3.7) G/W has exactly one minimal normal subgroup V/W contained in L/W; it is self-centralizing in L/W and complemented by M/W. Since M is f-abnormal for L, by (2.3.6) V/W is f-eccentric for L, and therefore $L/V = L/C_L(V/W) \notin f(p)$. Hence, writing $C = C_p(L)$, we have $C \nleq V$ and therefore $V < CV$. Let U/V be a minimal normal subgroup of G/V contained in RV/V. Since V/W is self-centralizing in L, $p \nmid |U:V|$, and therefore U/V is a q-chief factor of G for some $q \neq p$. From the isomorphism $G/V \underset{M}{\cong} M/W$ it follows that $U \cap M/W$ is a q-chief factor of M, but by the uniqueness of the minimal normal subgroup V/W we have $U \cap M \ntrianglelefteq G$ and therefore $N_G(U \cap M) = M$. Now $U \cap M/W$ is a Sylow p-complement of U/W, and since $T^p \leq M$, T^pW/W is a Sylow p-complement of $CW \cap M/W = (C \cap M)W/W$. Therefore $U \cap M \leq T^pW \cap U$, and since $p \nmid |T^pW \cap U:W|$ we have $U \cap M = T^pW \cap U$. Hence $N = N_G(T^p) \leq N_G(T^pW \cap U) = N_G(U \cap M) = M$ as required.

<u>3.1.6</u> <u>LEMMA</u>. Under the hypotheses of Lemma 3.1.5 if $\bar{N}$ is the $f(p)$ - normalizer of $L \cap M$ relative to M corresponding to L^p, then $N \leq \bar{N}$.

<u>Proof</u>. Write $C = C_p(L)$ and $\bar{C} = C_p(L \cap M)$. Since by hypothesis M is f-abnormal for L, we have $MC = G$, and therefore $(L \cap M)C = L \cap MC = L$. Hence $L \cap M/C \cap M \cong L/C \in f(p)$, and since $L \cap M/\bar{C} \in f(p)$ we therefore have $L \cap M/C \cap \bar{C} \in R_o f(p) = f(p)$. Since L^p is a Sylow p-complement of $L \cap M$, it follows that $\bar{T}^p = L^p \cap \bar{C}$ is the $f(p)$ - complement of $L \cap M$ corresponding to L^p. Moreover, because T^p is a Sylow p-complement of $C \cap M$, $T^p \cap \bar{C}$ is a Sylow p-complement of $C \cap M \cap \bar{C} = C \cap \bar{C}$. Since $T^p \cap \bar{C} \leq L^p \cap \bar{C} = \bar{T}^p$, we therefore have $T^p \cap \bar{C} = \bar{T}^p \cap C$. Now by (3.1.4) we have $\bar{N} = N_M(\bar{T}^p) = N_M(\bar{T}^p \cap C \cap \bar{C})$ $= N_M(T^p \cap \bar{C})$. Since $N \leq M$ by (3.1.5), we have $N \leq N_M(T^p \cap \bar{C}) = \bar{N}$ as required.

In the case $f(p) = 1$ we have $T^p = \bar{T}^p$ in (3.1.6) and therefore evidently $N = \bar{N}$. That we do not in general have equality here is shown by Example 6.1.4. The group G of that example has an $f(3)$ - normalizer of order 2.5 whereas the f-abnormal maximal subgroup M of G of index 3 has an $f(3)$ - normalizer of order $2.3^4.5$.

<u>3.2</u> In this section we introduce the concept of maximal subgroups and chief factors '$f(p)$ - critical for L', show the connection between them and discuss an important property.

<u>3.2.1</u> <u>DEFINITION</u>. A maximal subgroup M of G is called <u>$f(p)$ - critical for L</u> if it satisfies

(i) M is p-maximal in G ,

(ii) M is f-abnormal for L , and

(iii) $M\, O_{p'p}(L) = G$.

Since every p-maximal subgroup of G contains J_p (in the notation of section 2.4), it follows from the definition and (2.3.6) that G has a maximal subgroup $f(p)$ - critical for L if and only if G has a chief factor between J_p and $O_{p'p}(L)$ which is f-eccentric for L . Hence by (2.4.5) we have

3.2.2 THEOREM. G has maximal subgroups $f(p)$ - critical for L if and only if $L \notin \mathfrak{F}_p$.

3.2.3 DEFINITION. A chief factor H/K of G is called $f(p)$ - critical for L if it satisfies

(i) H/K is a complemented p-chief factor of G ,

(ii) H/K is f-eccentric for L , and

(iii) every p-chief factor of G below K is either f-central for L or not complemented.

Our next theorem shows a close connection between Definitions 3.2.1 and 3.2.3 .

3.2.4 THEOREM. A maximal subgroup of G is $f(p)$ - critical for L if and only if it complements a chief factor of G $f(p)$ - critical for L .

Proof. Let M be a maximal subgroup of G $f(p)$ - critical for L . As in the proof of (2.4.5) we may write $F(\bar{L}) = \bar{N}_1 \times \dots \times \bar{N}_r$ where

$\bar{L} = L/J$, etc. Since $F(\bar{L}) = O_{p'p}(L)/J$ and by hypothesis $M\,O_{p'p}(L) = G$, we have $\bar{N}_i \nleq \bar{M}$ $(= M/J)$ for some i, $1 \leq i \leq r$. In this case it is clear that N_i/J is a chief factor of G $f(p)$ - critical for L , for as it is a p-chief factor complemented by M by (2.3.6) it is f-eccentric for L and certainly no p-chief factors of G below J are complemented.

Conversely suppose M complements a chief factor H/K of G which is $f(p)$ - critical for L . For brevity write $F = O_{p'p}(L)$. Since H/K is a p-chief factor of G f-eccentric for L , by (2.3.6) M is a p-maximal subgroup of G f-abnormal for L so that conditions (i) and (ii) of (3.2.1) are satisfied. To complete the proof we now use induction on G to show $MF = G$. First suppose $K = 1$ so that H is a minimal normal subgroup of G . Since H/1 is f-eccentric for L , H is necessarily contained in L , for otherwise L would centralize it. As a normal p-subgroup of L, H is contained in F , and therefore $G = MH = MF$ as required. Therefore we may assume that G has a minimal normal subgroup N contained in K . If $N \leq J$ or $N \cap L = 1$, then $O_{p'p}(LN/N) = FN/N$ and again by induction we reach the desired conclusion $G = MFN = MF$. Hence we assume N/1 is a complemented p-chief factor of G and is therefore by hypothesis f-central for L . Writing $C = C_L(N)$ we have $L/C \in f(p)$. Define $\bar{F}$ by $\bar{F}/N = O_{p'p}(L/N)$ so that by induction we have $M\bar{F} = G$. By the remark after (2.4.5) F is the intersection of the centralizers in L of the p-chief factors of a given chief series of G , and therefore $\bar{F} \cap C = F$. We suppose $\bar{F} \cap C \leq M$ and get a contradiction. On this assumption M complements some chief factor R/S of G with $\bar{F} \cap C \leq S < R \leq \bar{F}$, and by (2.3.6) R/S is f-eccentric for L. Now $[C,\bar{F}] \leq \bar{F} \cap C$, and hence $C \leq C_L(\bar{F}/\bar{F} \cap C) \leq C_L(R/S)$. Thus $Aut_L(R/S)$ is isomorphic to a quotient group of $L/C \in f(p)$, and hence R/S is f-central for L. This contradiction shows $G = M(\bar{F} \cap C) = MF$.

<u>3.2.5</u> <u>THEOREM</u>. Let M be a p-maximal subgroup of G which supplements $O_{p'p}(L)$, and let H/K be a p-chief factor of G covered by M. If N is a normal subgroup of G containing $O_{p'p}(L)$, then writing $\bar{N} = N \cap M$ we have

$$Aut_N(H/K) \cong Aut_{\bar{N}}(H \cap M / K \cap M) .$$

<u>Proof</u>. Write $R = O_{p'p}(L)$. By the remark following (2.4.5) $R \leq C = C_N(H/K)$. By hypothesis we have $MR = G$ and therefore $\bar{N}R = (N \cap M)R = N \cap MR = N$. Hence $\bar{N}C = N$, and thus $N/C \cong \bar{N}/\bar{N} \cap C = N/C_{\bar{N}}(H/K)$. The result will follow if we can show H/K and $H \cap M/K \cap M$ are M-isomorphic; for they are then certainly $\bar{N}$-isomorphic since $\bar{N} < M$, and we shall therefore have $Aut_N(H/K) \cong Aut_{\bar{N}}(H/K) \cong Aut_{\bar{N}}(H \cap M/K \cap M)$ as desired. Hence we must show that $C_M(H/K) = C_M(H \cap M/K \cap M)$. Let $h \in H$; since by hypothesis $K(H \cap M) = H$, we may write $h = \bar{k}\,\bar{h}$ with $\bar{k} \in K$ and $\bar{h} \in H \cap M$. Let $m \in M$; then $h^m h^{-1} = m^{-1} \bar{k}\,\bar{h}\, m\, \bar{h}^{-1} \bar{k}^{-1} = k'(\bar{h}^m \bar{h}^{-1})\bar{k}^{-1}$ where $k' = \bar{k}^m \in K$. Thus $h^m h^{-1} \in K$ if and only if $\bar{h}^m \bar{h}^{-1} \in K$. Since $\bar{h}^m \bar{h}^{-1} \in M$ it follows that $Kh^m = Kh$ for all $h \in H$ if and only if $(K \cap M)\,\bar{h}^m = (K \cap M)\,\bar{h}$ for all $\bar{h} \in H \cap M$ which proves the desired result.

We state explicitly a result contained in the above proof.

3.2.6 THEOREM. Under the hypotheses of (3.2.5) we have

$$C_{\bar{N}}(H \cap M/K \cap M) = \bar{N} \cap C_G(H/K)$$

By specializing (3.2.5) and taking N equal to G and L in turn we have

3.2.7 COROLLARY. Under the hypotheses of (3.2.5) $H \cap M / K \cap M$ is a p-chief factor of M, and if H/K is f-central (f-eccentric) for L then $H \cap M / K \cap M$ is f-central (f-eccentric) for $\bar{L} = L \cap M$.

If we now take M to be a subgroup $f(p)$ - critical for L in the above corollary we get an indication of why this concept introduced in (3.2.1) proves useful; such a subgroup 'casts off' one p-chief factor of G f-eccentric for L while preserving the embedding of the remaining p-chief factors. Consequently we are able to eliminate one by one those p-chief factors of G f-eccentric for L by means of a descending chain of subgroups each maximal in the next.

3.3 We now have at our disposal most of the results we need to prove the important properties of $f(p)$ - normalizers.

3.3.1 THEOREM. A relative $f(p)$ - normalizer of L avoids the p-chief factors of G which are f-eccentric for L and covers all other chief factors. In particular, G is itself a relative $f(p)$ - normalizer of L if and only if $L \in \mathfrak{F}_p$.

Proof. Although this theorem may be deduced directly from Theorems 6.2

and 7.2 of [13], as an illustration of the concepts introduced in section 3.2 we give here an alternative proof by induction on $|L|$. If $L \in \mathfrak{F}_p$, by (2.4.6) every p-chief factor of G is f-central for L; hence $C_p(L) = O_{p'p}(L)$, and therefore $N_G(T^p) = G$. In this case the theorem is true and we have a starting point for the induction. We therefore assume $L \notin \mathfrak{F}_p$ and that the theorem has already been proved for all groups in which the soluble normal subgroup under consideration has order less than $|L|$. Let $N = N_G(T^p)$ be the f(p) - normalizer of L relative to G corresponding to L^p. By (3.2.2) G has an f(p) - critical maximal subgroup M for L, and this may be chosen to contain L^p. By (3.2.6) $M \cap C_p(L)$ is the intersection of the centralizers in M of those p-chief factors of M which are f-central for $\bar{L} = L \cap M$, and therefore T^p is the f(p) - complement of $L \cap M$ corresponding to L^p. By (3.1.5) we have $N = N_M(T^p)$, and since $|\bar{L}| < |L|$, by induction N covers all chief factors of M except those which are p-chief factors f-eccentric for $\bar{L}$, and these it avoids. If H/K is a p-chief factor of G f-eccentric for L, then either M avoids H/K, or by (3.2.7) $H \cap M/K \cap M$ is a p-chief factor of M f-eccentric for $\bar{L}$. In the second instance N avoids $H \cap M/K \cap M$, and in either case N avoids H/K. A further application of (3.2.7) to the remaining chief factors in a given chief series of G shows by considerations of order that N has the desired properties. (We observe that none of the results used in this proof makes use of (3.1.4), and so the proof is independent of [13].) The last assertion of the theorem follows from (2.4.6).

As an immediate consequence of (3.3.1) we have

3.3.2 COROLLARY. The product of the orders of those p-chief factors in a given chief series of G which are f-eccentric for L is equal to the index in G of a relative $f(p)$ - normalizer of L, and is therefore also equal to the number of $f(p)$ - complements of L.

Our next result also leans heavily on (3.3.1).

3.3.3 THEOREM. Let M be a p-maximal subgroup of G which supplements $O_{p'p}(L)$ and contains L^p. If $N = N_G(T^p)$ is the $f(p)$ - normalizer of L relative to G corresponding to L^p, then $M \cap N$ is the corresponding $f(p)$ - normalizer of $\bar{L} = L \cap M$ relative to M. In particular, if M is $f(p)$ - critical for L, an $f(p)$ - normalizer of $L \cap M$ relative to M is an $f(p)$ - normalizer of L relative to G.

Proof. Write $R = O_{p'p}(L)$. Since $R/O_{p'}(L) \in \mathfrak{N}$, by (2.3.8) G has a chief series $1 = G_0 < G_1 < \ldots < G_{i-1} < G_i < \ldots < G_r = G$, with $G_{i-1} = R \cap M$ and $G_i = R$. Write $C = C_L(R/R \cap M)$ if $R/R \cap M$ is f-central for L and $C = L$ otherwise. Let C^* denote the intersection of the centralizers in L of all those p-chief factors except G_i/G_{i-1} in the above chief series which are f-central for L. Then $C_p(L) = C^* \cap C$, and taking $N = L$ in (3.2.7) we have $C_p(\bar{L}) = \bar{L} \cap C^*$. Since $L^p \leq \bar{L}$, we therefore have $C^* \cap L^p = C_p(\bar{L}) \cap L^p$, $= \bar{T}^p$ say. Now whatever normalizes $C^* \cap L^p$ also normalizes $C \cap C^* \cap L^p = C_p(L) \cap L^p = T^p$, and writing $\bar{N} = N_M(\bar{T}^p)$ we therefore have $\bar{N} = M \cap N_G(C^* \cap L^p) \leq M \cap N$. However, by (3.3.1) $M \cap N$ avoids those p-chief factors of

M which are f-eccentric for $\bar{L}$ and $\bar{N}$ covers those which are f-central for $\bar{L}$. Hence $|\bar{N}| \geq |M \cap N|$ and therefore $\bar{N} = N \cap M$ as required.

Theorem 3.3.1 also plays a part in proving the following important result.

3.3.4 THEOREM. If N is a relative $f(p)$ - normalizer of L, and R any normal subgroup of G, then NR/R is an $f(p)$ - normalizer of LR/R relative to G/R. In other words, relative $f(p)$ - normalizers are homomorphism - invariant.

Proof. Let $N = N_G(T^p)$ be an $f(p)$ - normalizer of L relative to G (corresponding to L^p, say). It is clear that $C_p(LR/R) \geq C_p(L)R/R$ and that $NR \leq N_G(T^pR)$. Now $\bar{T}^p/R = L^pR/R \cap C_p(LR/R)$ is a Sylow p-complement of $C_p(LR/R)$. Therefore $(\bar{T}^p \cap C_p(L)R)/R$ is a Sylow p-complement of $C_p(L)R/R$, and we have $T^pR/R = (\bar{T}^p \cap C_p(L)R)/R$. Since by (3.1.4) $N_G(\bar{T}^p \cap C_p(L)R) = N_G(\bar{T}^p)$, we have $NR/R \leq N_G(T^pR)/R \leq N_{G/R}(\bar{T}^p/R)$, the $f(p)$ - normalizer of LR/R relative to G/R corresponding to L^pR/R. Now a p-chief factor $(H/R)/(K/R)$ of G/R is f-eccentric for LR/R if and only if H/K is f-eccentric for L, because $R \leq C_G(H/K)$ and hence $LR/C_{LR}(H/K) = L/C_L(H/K)$. Therefore by the covering and avoidance property of (3.3.1) we have $|G:NR| = |G/R : N_{G/R}(\bar{T}^p/R)|$ and the theorem now follows.

3.3.5 THEOREM. Let $N = N_G(T^p)$ be the $f(p)$ - normalizer of L relative to G corresponding to L^p, and let H/K be a p-chief factor of G covered by N. Then we have

(a) $Aut_G(H/K) \cong Aut_N(H \cap N/K \cap N)$ so that $H \cap N/K \cap N$ is a chief factor of N, and

(b) $\mathrm{Aut}_L(H/K) \cong \mathrm{Aut}_{\bar{L}}(H \cap N/K \cap N)$, where $\bar{L} = L \cap N$.

Proof. We use induction on $|L|$. If $L \in \mathfrak{F}_p$, then $N = G$ and assertion (a) is trivially true. We may therefore suppose $L \notin \mathfrak{F}_p$, and that (a) holds for all groups G in which the soluble normal subgroup under consideration has order smaller than $|L|$. Let M be a maximal subgroup of G $f(p)$ - critical for L containing L^p . By (3.1.5) $N \leq M$; hence H/K is covered by M and by (3.2.5) $\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_M(H \cap M/K \cap M)$. Therefore $H \cap M/K \cap M$ is a p-chief factor of M covered by N , and by (3.3.3) N is an $f(p)$ - normalizer of $L \cap M$ relative to M ; hence, as $|L \cap M| < |L|$, by induction we have $\mathrm{Aut}_M(H \cap M/K \cap M) \cong \mathrm{Aut}_N(H \cap M \cap N/K \cap M \cap N) = \mathrm{Aut}_N(H \cap N/K \cap N)$. This proves assertion (a). The proof of (b) follows from (3.2.5) in the same way.

3.3.6 THEOREM. In the notation of (3.3.5) the soluble normal subgroup $L \cap N$ of N belongs to $\mathfrak{F}_p$.

Proof. Let H/K be a p-chief factor of G . If H/K is f-eccentric for L , by (3.3.1) H/K is avoided by N and we have $H \cap N = K \cap N$. On the other hand, if H/K is f-central for L , then $H \cap N/K \cap N$ is a p-chief factor of N f-central for $L \cap N$ by (3.3.5). Therefore the intersection of N with a chief series of G is a normal series of N in which every factor whose order is divisible by p is a chief factor f-central for $L \cap N$. The result now follows from (2.4.6).

3.4 In this final section of chapter three we give two characterizations of relative $f(p)$- normalizers in terms of maximal chains.

This is followed by a brief discussion of how our earlier results specialize to the case of absolute $f(p)$ - normalizers when $L = G$, and we conclude with a theorem connecting the relative and absolute $f(p)$ - normalizers when G itself is soluble.

<u>3.4.1</u> <u>DEFINITIONS</u>. If P is a property specifying the way a subgroup is embedded in a group - for example, P could be the property of being 'maximal', 'normal', etc. - we say a chain of subgroups $X_r \leq X_{r-1} \leq \ldots \leq X_0 = G$ is a <u>P chain</u> if X_i is a P subgroup of X_{i-1} for $i = 1, 2, \ldots ,r$. If P is a relative property in the sense that G may have subgroups 'P' for L, (e.g. f-abnormal for L), by a <u>relative P chain</u> we mean one in which X_i is a subgroup of X_{i-1} P for $L \cap X_{i-1}$.

<u>3.4.2</u> <u>THEOREM</u>. Let $G_r < G_{r-1} < \ldots < G_0 = G$ be a relative $f(p)$ - critical maximal chain of G and suppose G_r is minimal in the sense that G_r contains no subgroups $f(p)$ - critical for $L \cap G_r$; then G_r is a relative $f(p)$ - normalizer of L in G. Moreover, every relative $f(p)$ - normalizer is expressible as the minimal member of such a chain.

<u>Proof</u>. Since $|G:G_r|$ is a power of the prime p, so is $|L:L \cap G_r|$, and therefore G_r contains a Sylow p-complement L^p of L. If we repeatedly apply the last part of Theorem 3.3.3 moving step by step down the chain, we deduce that $N = N_G(C_p(L) \cap L^p)$ is an $f(p)$ - normalizer of $L \cap G_r$ relative to G_r. But since G_r has no maximal subgroups $f(p)$ - critical for $L \cap G_r$, by (3.2.2) $L \cap G_r \in \mathfrak{F}_p$,

and therefore by the last statement of (3.3.1) we have $N = G_r$. The last remark in the statement of this theorem follows at once from the conjugacy of relative $f(p)$ - normalizers.

3.4.3 THEOREM. Let $\mathcal{C}$ be the set of subgroups which can be joined to G by a relative f-abnormal p-maximal chain. Then the minimal members of $\mathcal{C}$ are precisely the relative $f(p)$ - normalizers of L .

Proof. Let X be a minimal member of $\mathcal{C}$. By repeated application of (3.1.6) we see that X contains an $f(p)$ - normalizer N of L relative to G . But by (3.4.2) N itself belongs to $\mathcal{C}$, and therefore by minimality we have $X = N$. It follows from their conjugacy that every relative $f(p)$ - normalizer occurs in this way.

To recapitulate let us now consider what happens when $L = G$. In this case $C_p(G)$ is the intersection of the centralizers of the f-central p-chief factors of G . If S^p is a Sylow p-complement of G , then $N = N_G(C_p(G) \cap S^p)$ is the (absolute) $f(p)$ - normalizer of G corresponding to S^p . The set of all such N forms a characteristic conjugacy class of abnormal subgroups of G each of which avoids the f-eccentric p-chief factors of G and covers the rest. Each N belongs to the p-local formation $\mathfrak{F}_p$, and a homomorphism of G maps N into an $f(p)$ - normalizer of the homomorphic image of G . If $G \in \mathfrak{F}_p$, then $N = G$; on the other hand if $G \notin \mathfrak{F}_p$, G has an f-abnormal p-maximal subgroup M supplementing $O_{p'p}(G)$, and such M are called the $f(p)$ - critical maximal subgroups of G . Every such M contains some $f(p)$ - normalizer N of G such that N is also an $f(p)$ - normalizer of M . The

$f(p)$ - normalizers of G are precisely the minimal members of the $f(p)$ - critical maximal chains of G. They may also be characterized abstractly as the minimal members of the set of those subgroups which can be joined to G by an f-abnormal p-maximal chain.

Our next result gives a connection between relative and absolute $f(p)$ - normalizers.

<u>3.4.4</u> <u>THEOREM</u>. Let L be a normal subgroup of the soluble group G, and let $N = N_G(T^p)$ be the $f(p)$ - normalizer of L relative to G corresponding to L^p; further suppose that the formation $f(p)$ is Sn-closed where Sn is the closure operation defined by: $G \in \mathrm{Sn}\mathfrak{X} \Leftrightarrow G$ is a subnormal subgroup of an $\mathfrak{X}$-group. Then an absolute $f(p)$ - normalizer of N is an absolute $f(p)$ - normalizer of G.

<u>Proof</u>. We use induction on $|G|$. Let H/K be an f-central p-chief factor of G. Writing $C = C_G(H/K)$ we have $L/C_L(H/K) = L/L \cap C \cong LC/C \in \mathrm{Sn}(G/C) \leq \mathrm{Sn}\, f(p) = f(p)$ by hypothesis, and therefore H/K is f-central for L. If $L \in \mathfrak{F}_p$, then $N = G$ and the theorem is true. If $L \notin \mathfrak{F}_p$, then by (3.3.3) N is the $f(p)$ - normalizer of $L \cap M$ relative to M where M is a maximal subgroup of G which is $f(p)$ - critical for L and contains L^p. Since M is f-abnormal for L, it follows from (2.3.6) and the above remarks that M is f-abnormal in G; because $M\, O_{p'p}(G) \geq M\, O_{p'p}(L) = G$, M is therefore an $f(p)$ - critical maximal subgroup of G. Hence by (3.3.3) an $f(p)$ - normalizer of M is an $f(p)$ - normalizer of G. But since $|M| < |G|$, by induction an $f(p)$ - normalizer of N is an $f(p)$ - normalizer of M, and the proof is complete.

In conclusion we give an example to show to show (3.4.4) is in general no longer true when we drop the restriction $f(p) = Sn\, f(p)$ from the hypotheses.

3.4.5 EXAMPLE. Let $f(2)$ be the formation comprising all groups X which satisfy

(a) X has an elementary Abelian normal 3-subgroup Y (possibly trivial) such that X/Y is an elementary Abelian 2-group , and

(b) X has no central 3-chief factors.

It is easy to see that $f(2)$ is the smallest formation containing the symmetric group Σ_3 , and that it does not contain the cyclic group of order 3. Let $G = \Sigma_4$ and $L = A_4 \triangleleft \Sigma_4$. The only 2-chief factor $E_4/1$ of L has $\mathrm{Aut}_L(E_4/1) = 3$ and is therefore f-eccentric. Hence $C_2(L) = L$. Using the cycle notation for Σ_4, $T^2 = \langle (123) \rangle$ is therefore an $f(2)$ - complement of L , and $N_G(T^2) = \langle (123), (12) \rangle$ has order 6. However, G is its own $f(2)$ - normalizer, and therefore the conclusion of Theorem 3.4.4 does not hold in this case.

Chapter Four

f - NORMALIZERS

4.1 We recall that L denotes a soluble normal subgroup of an arbitrary group G, and f a formation function defining $\mathfrak{F}$ locally.

4.1.1 DEFINITIONS. Let $\mathfrak{S}$ be a Sylow system of L, and let T^p be the $f(p)$ - complement of L corresponding to $L^p \in \mathfrak{S}$ - we recall that $T^p = L^p \cap C_p(L)$. We call the set $\{ T^p \mid p \mid |L| \}$ the complete set of f(p) - complements corresponding to $\mathfrak{S}$, and the set $\mathfrak{J}$ of 2^r (not necessarily distinct) subgroups contained in the lattice of intersections of $\{ T^p \}$ we call the f-system of L corresponding to $\mathfrak{S}$, ($r = |\sigma(L)|$). It is clear from (3.1.2) that $\mathfrak{J}$ depends on f, $\mathfrak{S}$ and L, but not on the group G. The subgroup

$$D = N_G(\mathfrak{J}) = \bigcap_{p \mid |L|} N_G(T^p)$$

is called the f - normalizer of L relative to G corresponding to $\mathfrak{S}$, or simply a relative f - normalizer of L when G and $\mathfrak{S}$ are understood. When $L = G$ we call D an absolute f - normalizer of G. When $f(p) = 1$ for all primes p an f-system becomes a Sylow system, and therefore D a system normalizer, (see P. Hall, [12] and [13]). Some but by no means all of P. Hall's classical results for Sylow systems carry over to the more general f-systems. It is clear from the corresponding results for Sylow systems that any two f-systems of L are conjugate in L, and since $C_p(L) \triangleleft L$, it follows that the set of f-systems is invariant under automorphisms of L; if $L \triangleleft G$, this set is also invariant under automorphisms of G. It is also true that the members of an f-system are pairwise permutable. This is a consequence of

4.1.2 LEMMA. If S and S' are permutable Hall subgroups of L, and if H and K are normal subgroups of L, then $(H \cap S) \perp (K \cap S')$.

Proof. Let $x \in \bar{H} = H \cap S$ and $y \in \bar{K} = K \cap S'$. Then $x\,y = y[y,x^{-1}]x$. But $[y,x^{-1}] \in [K,H] \leq H \cap K$, $= N$ say, since H and K are normal. Also $[y,x^{-1}] \in SS'$ and so $[y,x^{-1}] \in N \cap SS' = (N \cap S)(N \cap S')$ since $N \cap S$, $N \cap S'$, and $N \cap SS'$ are Hall subgroups of N. Therefore $[y,x^{-1}] = \bar{y}\,\bar{x}$ with $\bar{y} \in N \cap S' \leq \bar{K}$ and $\bar{x} \in N \cap S \leq \bar{H}$. Hence $x\,y = y\,\bar{y}\,\bar{x}\,x \in \bar{K}\bar{H}$. Since x and y were chosen arbitrarily, we have $\bar{H}\bar{K} \leq \bar{K}\bar{H}$. Hence $\bar{H} \perp \bar{K}$ as required.

Now let X and Y be members of the f-system $\mathfrak{J}$ which corresponds to the Sylow system $\mathfrak{S}$. Then $X = \bigcap_i (C_{p_i}(L) \cap L^{p_i}) = (\bigcap_i C_{p_i}(L)) \cap (\bigcap_i L^{p_i})$ for suitable primes $p_i \,\big|\, |L|$, so that X is of the form $H \cap S$ with $H \triangleleft L$ and $S = \bigcap_i L^{p_i} \in \mathfrak{S}$. Y is likewise of the same form, and since the members of $\mathfrak{S}$ are pairwise permutable Hall subgroups of L, our contention that $X \perp Y$ follows at once from (4.1.2).

A Sylow system is always generated by a complete set of pairwise permutable Sylow subgroups and the corresponding system normalizer is then the intersection of the normalizers of these Sylow subgroups. We next show that neither of these properties carries over to f-systems and their normalizers.

4.1.3 EXAMPLE. Let M and N be elementary Abelian groups of order

2^3 and 11^2 respectively. Aut(M) , the simple group of order 168 , contains an irreducible subgroup K isomorphic with the non-Abelian group of order 21 , and K has a normal subgroup $\bar{K}$ of order 7 . $K/\bar{K}$ is cyclic of order 3 , and is faithfully, irreducibly represented as a group of automorphisms of N . K may be identified with a subgroup of $\mathrm{Aut}(M \times N)$, acting on M as the group of order 21 , and on N as the group of order 3 with $\bar{K}$ acting trivially. Let H be the splitting extension of $M \times N$ by K , and take f(p) to be the class of Abelian groups of exponent $p - 1$ so that by (2.2.7) f is an S-closed formation function defining $\mathfrak{U}$ locally. Since H is clearly soluble, we put $H = L = G$ in the standard notation of this chapter, and proceed to exhibit an f-system of G and its absolute normalizer. Let K^* be a complement of $\bar{K}$ in K , and let $\mathfrak{S}$ be the Sylow system of G generated by the Sylow p-complements

$$\{ H^2 = KN \ , \quad H^3 = \bar{K}(M \times N) \ , \quad H^7 = K^*(M \times N) \ , \quad H^{11} = KM \} \ .$$

Then the corresponding f-system $\mathfrak{J}$ is generated by the f(p) - complements

$$\{ T^2 = KN \ , \quad T^3 = \bar{K}(M \times N) \ , \quad T^7 = (M \times N) \ , \quad T^{11} = KM \} \ ,$$

and K is evidently the normalizer of $\mathfrak{J}$. The p-groups belonging to $\mathfrak{J}$ are $\bar{K}$, M and N , and therefore as there is no 3-group in $\mathfrak{J}$ it cannot be characterized as the family of subgroups generated by its p-groups. Further, the intersection of the normalizers of the p-groups belonging to $\mathfrak{J}$ is KN which properly contains an f - normalizer of H . Finally we observe that the 2^4 formally distinct members of the lattice of intersections comprising $\mathfrak{J}$ are not actually distinct since $T^3 \cap T^7 = T^7$.

4.2 We now discuss some of the fundamental properties of f - normalizers which hold without any additional restrictions on f . We observe from (3.1.4) that our choice of an f-system is to some extent arbitrary, and that the alternative definition $T^p = L^p \cap R_{f(p)}(L)$ for the $f(p)$ - complements would have led to the same f - normalizer. From our discussion of the conjugacy of f-systems in section 4.1 we have at once

4.2.1 THEOREM. The f - normalizers of L relative to G form a characteristic conjugacy class of L and, in particular, they form a conjugacy class of G . If $L \triangleleft G$, the conjugacy class is characteristic in G .

Another basic result concerns the covering and avoidance properties.

4.2.2 THEOREM. An f - normalizer of L relative to G covers those chief factors of G which are f-central for L and avoids the rest. In particular, G is itself a relative f - normalizer of L if and only if $L \in \mathfrak{F}$.

Proof. Let D be an f - normalizer of L relative to G , so that, as the intersection of a complete set of relative $f(p)$ - normalizers of L, by (3.3.1) D certainly avoids all chief factors of G f-eccentric for L . Again by (3.3.1) $f(p)$ - normalizers have index a power of p in G , and therefore members of a complete set containing one for each prime p dividing L are pairwise permutable. Hence the index of D in G equals the product of the indices in G of the $f(p)$ - normalizers, and this in turn is equal to the product of the orders of those chief factors in a given chief series of G which are f-eccentric for L. The order of D is the product of the orders of its projections onto the chief factors of a given chief series of G, and therefore D covers the chief factors of G which are f-central for L. Thus G itself is a relative f - normalizer of L if and only if every chief factor of G is f-central for L, and by (2.4.7) this happens if and only if $L \in \mathfrak{F}$.

4.2.3 COROLLARY. If D is an f- normalizer of L relative to G then $|D|$ equals the product of the orders of those chief factors in a given chief series of G which are f-central for L. $|G:D| = |L:L \cap D|$, the number of f-systems of L, equals the product of the orders of those chief factors on a given chief series of G which are f-eccentric for L.

4.2.4 THEOREM. If D is a relative f - normalizer of L, and if θ is a homomorphism of G onto G^*, then $D^* = \theta(D)$ is an f - normalizer of L^* relative to G^*.

Proof. Let R be the kernal of θ; it is sufficient to prove the result for the natural homomorphism $G \to G/R$. Let $D = \bigcap_{p \,|\, |L|} N_p$ for suitable $f(p)$ - normalizers N_p of L relative to G. Then $DR/R \leq N_pR/R$ for each p. But by (3.3.4) N_pR/R is an $f(p)$ - normalizer of LR/R relative to G/R, and therefore DR/R is contained in $\bigcap_{p \,|\, |L|} N_pR/R$ which is a relative f - normalizer of LR/R. But there is a natural one-to-one correspondence between chief factors of G above R which are f-central (f-eccentric) for L, and chief factors of G/R

which are f-central (f-eccentric) for LR/R. Hence by the covering and avoidance property of (4.2.2) DR/R has the same order as an f-normalizer of LR/R relative to G/R and must therefore itself be one.

Our next result follows immediately from the defintion of a relative f-normalizer and the remarks in (3.1.3).

<u>4.2.5</u> <u>THEOREM</u>. The intersection with L of an f-normalizer of L relative to G is an absolute f-normalizer of L.

<u>4.2.6</u> <u>DEFINITION</u>. We say a normal subgroup N of G is <u>embedded f-centrally for L in G</u> if all the chief factors of G below N are f-central for L. Since the product of two normal subgroups of G embedded f-centrally for L is clearly again embedded f-centrally for L, we may form the product $Z_f(G:L)$ of all such normal subgroups. We call $Z_f(G:L)$ the <u>f-hypercentre of G for L</u>. It is evident from the covering and avoidance property of (4.2.2) that the normal interior (or core) in G of an f-normalizer of L relative to G, or equivalently the intersection of all the relative f-normalizers of L, is equal to the f-hypercentre of G for L. Hence by (4.2.4) we have

<u>4.2.7</u> <u>THEOREM</u>. If θ is a homomorphism of G onto G^*, the intersection of all the f-normalizers of $L^* = \theta(L)$ relative to G^* is the f-hypercentre of G^* for L^*.

We conclude this section with a necessary and sufficient condition for a maximal subgroup of G to contain a relative f-normalizer of L.

<u>4.2.8</u> <u>THEOREM</u>. A maximal subgroup of G contains an f - normalizer of L relative to G if and only if it is f-abnormal for L .

<u>Proof</u>. Suppose $M \lessdot G$ and that M contains a relative f - normalizer of L . M supplements some chief factor H/K of G , and since D covers chief factors f-central for L by (4.2.2), we must have H/K f-eccentric for L . By (2.3.6) M is therefore f-abnormal for L . Conversely, suppose M ($\lessdot$ G) is f-abnormal for L . Then by definition M supplements L in G , is p-maximal for some prime p, and therefore by (3.1.5) contains an f(p) - normalizer N_p of L relative to G . N_p in turn contains an f - normalizer of L relative to G , and the proof is complete.

<u>4.3</u> In order to extend more of the classical results for system normalizers to the general situation under consideration here, we need to impose certain restrictions on the formation function f . Throughout this section we therefore demand that f is integrated, for this will enable us to expand the earlier concept of f(p) - critical subgroup and factors, and later to develop chain characterizations of f - normalizers.

<u>4.3.1</u>. <u>DEFINITION</u>. A maximal subgroup M of G is called **<u>f-critical for L</u>** if it satisfies

(i) M is f-abnormal for L , and

(ii) M F(L) = G .

<u>4.3.2</u> <u>LEMMA</u>. $L \in \mathfrak{F}$ if and only if every minimal normal subgroup of $G/\emptyset(G) \cap L$ contained in $L/\emptyset(G) \cap L$ is f-central for L (when considered as a chief factor of G).

Proof. Write $L^* = \emptyset(G) \cap L$ and $\bar{G} = G/L^*$, etc. By (2.4.3) $F(\bar{L}) = F(L)/L^*$ and we may write $F(\bar{L}) = \bar{N}_1 \times \dots \times \bar{N}_r$, where $\bar{N}_i = N_i/L^*$ is a minimal normal subgroup of $\bar{G}$, $i = 1, 2, \dots ,r$. $\bar{N}_i$ is an irreducible G-module, and by Clifford's Theorem qua L-module decomposes into the direct product $M_{i1} \times \dots \times M_{i\,s_i}$ of L-irreducible components such that

$$C_L(\bar{N}_i) = \bigcap_{j=1}^{s_i} C_L(\bar{M}_{i\,j}) . \qquad (*)$$

First suppose $L \in \mathfrak{F}$. Then the $\bar{M}_{i\,j}$ are f-central chief factors of L , and so $L/C_L(\bar{M}_{i\,j}) \in f(p)$, where $p \mid |\bar{N}_i|$. Hence $L/C_L(\bar{N}_i) = L / \bigcap_{j=1}^{s_i} C_L(\bar{M}_{i\,j}) \in R_0 f(p) = f(p)$, and so each $\bar{N}_i$ is f-central for L .
Conversely, suppose N_i/L^* is f-central for L , $i = 1, 2, \dots ,r$. Then writing $\bar{C}_i = C_{\bar{G}}(\bar{N}_i)$ we have $\bar{L}/\bar{C}_i \in f(p)$, where $p \mid |\bar{N}_i|$. Since f is integrated, we have $\bar{L} / \bigcap_{i=1}^{r} \bar{C}_i \in R_0\mathfrak{F} = \mathfrak{F}$, and by the well-known fact that $C_{\bar{L}}(F(\bar{L})) \leq F(\bar{L})$ it follows that $\bar{L}/F(\bar{L}) \in Q\mathfrak{F} = \mathfrak{F}$. Moreover, L induces on $\bar{M}_{i\,j}$ a group of automorphisms isomorphic with a homomorphic image of $L/C_L(\bar{N}_i) \in f(p)$, so that every chief factor of $\bar{L}$ below $F(\bar{L})$ is f-central. Thus every chief factor of $\bar{L}$ is f-central and hence $\bar{L} \in \mathfrak{F}$. It now follows from (2.4.7) that $L \in \mathfrak{F}$ and the proof is complete.

Since the chief factors N_i/L^* (in the above notation) are complemented, and since their complements are maximal subgroups of G supplementing L , (4.3.2) together with (2.3.6) gives us

<u>4.3.3</u> <u>THEOREM</u>. G has maximal subgroups f-critical for L if and only if $L \notin \mathfrak{F}$.

<u>4.3.4</u> <u>DEFINITION</u>. A chief factor H/K of G is called <u>f-critical</u>

for L if it satisfies

(i) H/K is complemented and f-eccentric for L, and

(ii) every chief factor of G below K is either not complemented or f-central for L.

As with the similar concepts in chapter three we show there is a strong tie between f-critical chief factors and f-critical maximal subgroups.

4.3.5 THEOREM. A maximal subgroup of G is f-critical for L if and only if it complements a chief factor of G f-critical for L.

Proof. Let M be a maximal subgroup of G f-critical for L, and let $F(\bar{L}) = \bar{N}_1 \times \dots \times \bar{N}_r$, where $\bar{L} = L/L^*$, etc., as in the proof of (4.3.2). We have $M\,F(L) = G$, and since M is maximal, $M \geq L^* = L \cap \emptyset(G)$. Since $\bar{M}\,F(\bar{L}) = \bar{G}$, $\bar{M}$ cannot contain every $\bar{N}_i$, and therefore by (2.3.3) M must complement some chief factor N_j/L^* of G. Since M is f-abnormal for L, by (2.3.6) N_j/L^* is f-eccentric for L and therefore clearly f-critical for L.

Conversely, let M be a complement of a chief factor H/K of G f-critical for L. By (2.3.6) M satisfies requirement (i) of Definition 4.3.1, and so it remains to show that $M\,F(L) = G$. We write $J = F(L)$ and proceed by induction on $|G|$. If $K = 1$ then H is a minimal normal subgroup of G and either $H \cap L = 1$ or $H \leq L$. The first possibility would imply $[H,L] = 1$, and H would be central for L which is ruled out. Hence the second alternative prevails, $H \leq J$ and $MJ \geq MH = G$ as required. Hence we may assume that G

has a minimal normal subgroup $N \leq K$. If $N \leq \phi(G) \cap L$ or $N \cap L = 1$, then $F(LN/N) = JN/N$, and since $|G/N| < |G|$ by induction $G = MJN = MJ$. We therefore assume that $N/1$ is a complemented chief factor of G contained in L ; thus it is a p-chief factor for some prime p and by hypothesis it is f-central for L . Let $C = C_L(N)$ so that $L/C \in f(p)$. Let $J^*/N = F(L/N)$ so that by the remarks following (2.4.5) $J^* \cap C = J$. Suppose, for a contradiction, that $J^* \cap C \leq M$. Since $|G/N| < |G|$, by induction $J^*M = G$, and therefore M must complement some chief factor R/S of G such that $J^* \cap C \leq S < R \leq J^*$. But $[J^*,C] \leq J^* \cap C$ since both are normal in G , and therefore $C \leq C_L(R/S)$, whence R/S is f-central for L . But by (2.3.6), since M is f-abnormal for L , R/S is f-eccentric for L , and we have a contradiction. Hence $J = J^* \cap C \nleq M$, and therefore $MJ = G$ as required.

4.3.6 THEOREM. Let M be a maximal subgroup of G supplementing $F(L)$ and let H/K be a chief factor of G covered by M . If N is any normal subgroup of G containing $F(L)$ and if $\bar{N} = M \cap N$, then we have
$$\mathrm{Aut}_N(H/K) \cong \mathrm{Aut}_{\bar{N}}(H \cap M / K \cap M) .$$

Proof. Since $F(L)$ centralizes every chief factor of G , the proof of this result follows almost word for word that of (3.2.5) with $F(L)$ replacing $O_{p'p}(L)$ throughout.

Continuing the analogy with chapter three we have

4.3.7 THEOREM. With the same hypotheses as (4.3.6) we have
$$C_{\bar{N}}(H \cap M / K \cap M) = \bar{N} \cap C_G(H/K) .$$

4.3.8 COROLLARY. If M is a maximal subgroup of G supplementing $F(L)$ (in particular, if M is a maximal subgroup of G f-critical for L), and if M covers the chief factor H/K of G, then $H \cap M/K \cap M$ is a chief factor of M, and is f-central for $L \cap M$ if and only if H/K is f-central for L.

From this with the help of (2.4.7) we deduce

4.3.9 COROLLARY. If $L \in \mathfrak{F}$, and M is a maximal subgroup of G supplementing $F(L)$, then $L \cap M \in \mathfrak{F}$.

The usefulness of the concept of a maximal subgroup 'f-critical for L' is that it enables us to cast off one f-eccentric chief factor of G, at the same time preserving the automorphism groups induced by the whole group and by the inherited soluble normal subgroup on the remaining chief factors.

4.4 In this section, except where otherwise stated, we continue to assume that $\mathfrak{F}$ is defined locally by the integrated formations $\{ f(p) \}$, for this will enable us later to capitalize on the results of section 4.3.

4.4.1 THEOREM. Let D be the normalizer in G of the f-system $\mathfrak{I}$ corresponding to the Sylow system $\mathfrak{S}$ of L. Suppose $L \notin \mathfrak{F}$, and let M be a maximal subgroup of G f-abnormal for L such that $\mathfrak{S}$ reduces into $L \cap M$, $= \bar{L}$ say. If $\bar{D}$ is the normalizer in M of the f-system $\bar{\mathfrak{I}}$ of $\bar{L}$ corresponding to the Sylow system $\mathfrak{S} \cap \bar{L}$, then $D \leq \bar{D}$.

Proof. Suppose $\mathfrak{J}$ is generated by the complete set of $f(r)$ - complements $\{T^r\}$, $\bar{\mathfrak{J}}$ by $\{\bar{T}^r\}$ and let p be the prime dividing $|G:M|$. By (3.1.6) $D \leq N_G(T^p) \leq N_M(\bar{T}^p) \leq M$, and therefore it is sufficient to prove that if $q \neq p$, then $M \cap N_G(T^q) \leq N_M(\bar{T}^q)$. Now $L/C_q(L) \in f(q) \leq \mathfrak{J}$ since the formations are integrated, and therefore every chief factor of G above $C_q(L)$ is f-central for L. Hence by (2.3.6), since M is f-abnormal for L, we have $C_q(L) \nleq M$, and therefore

$$M\, C_q(L) = G . \qquad (*)$$

Apart from this equation we make no further use of the fact that f is integrated in the rest of the proof. From (*) it follows that $\bar{L}\, C_q(L) = L$; for we have $\bar{L}\, C_q(L) = (L \cap M)\, C_q(L) = L \cap M\, C_q(L) = L$. Therefore $\bar{L}/C_q(L) \cap M \cong L/C_q(L) \in f(q)$. Now $|L:\bar{L}| = |G:M|$ is a power of the prime p, and therefore $T^q \cap \bar{L} = (L^q \cap C_q(L)) \cap (M \cap L) = (L^q \cap \bar{L}) \cap (C_q(L) \cap \bar{L})$ is a q-complement of $C_q(L) \cap \bar{L}$, because $L^q \cap \bar{L}$ is a q-complement of $\bar{L}$ and $C_q(L) \cap \bar{L} \lhd \bar{L}$. Hence $T^q \cap C_q(\bar{L})$ is a q-complement of $C_q(L) \cap C_q(\bar{L}) < C_q(L) \cap \bar{L}$. However from above we have $\bar{L}/C_q(L) \cap C_q(\bar{L}) = \bar{L}/C_q(L) \cap M \cap C_q(\bar{L}) \in R_0 f(q) = f(q)$, and therefore by (3.1.4) we have $N_M(\bar{T}^q) = N_M(T^q \cap C_q(\bar{L}))$. Hence as $C_q(\bar{L}) \lhd M$ we have $M \cap N_G(T^q) = N_M(T^q) \leq N_M(T^q \cap C_q(\bar{L})) = N_M(\bar{T}^q)$ as desired.

Our next three results are valid in general without the restriction that f is integrated.

<u>4.4.2</u> <u>THEOREM</u>. Let D be the normalizer in G of the f-system $\mathfrak{J}$ corresponding to the Sylow system $\mathfrak{S}$ of L. If M is a maximal subgroup of G which supplements $F(L)$ such that reduces into $L \cap M$, then $D \cap M$ is the f - normalizer of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$.

<u>Proof</u>. Let $\bar{\mathfrak{J}}$ be the f-system of $L \cap M$ corresponding to $\mathfrak{S} \cap M$, and suppose M is p-maximal in G. Since $F(L) \leq O_{p'p}(L)$ we may apply (3.3.3) to obtain $N_G(T^p) \cap M = N_M(\bar{T}^p)$. Now suppose $q \neq p$. Since $F(L) \leq C_q(L)$, condition (*) in the proof of (4.4.1) is satisfied, and therefore it follows from the conclusion of that part of the proof that

$$N_G(T^q) \cap M \leq N_M(\bar{T}^q) . \qquad (+)$$

Now $|G:N_G(T^q)|$ is a power of q, and therefore it is equal to $|M:N_G(T^q) \cap M|$ since $G = N_G(T^q)M$. Now by (3.3.2) $|M:N_M(\bar{T}^q)|$ is the product of the orders of those chief factors in a given chief series of M which are f-eccentric for $L \cap M$, and by (4.3.8) this has the same value as the similar quantity $|G:N_G(T^q)|$ for G. Hence the two subgroups on either side of the inequality (+) have the same index in M, and are therefore equal. Hence $D \cap M = \bigcap_{r||L|}(N_G(T^r) \cap M) = \bigcap_{r||L|} N_M(\bar{T}^r) = N_M(\bar{\mathfrak{J}})$, as required.

Repeated application of (4.4.2) gives

<u>4.4.3</u> <u>COROLLARY</u>. Let D be the normalizer in G of the f-system $\mathfrak{J}$ corresponding to the Sylow system $\mathfrak{S}$ of L, and let X be joined to G by a maximal chain of the form $X = X_r < X_{r-1} < \dots < X_0$

$X_0 = G$, where $X_i\, F(L \cap X_{i-1}) = X_{i-1}$ and $\mathfrak{S}$ reduces into $L \cap X_i$, $i = 1, 2, \ldots, r$. Then $D \cap X$ is the f - normalizer of $L \cap X$ relative to X corresponding to $\mathfrak{S} \cap X$. In particular, this conclusion holds if X is a supplement of $F(L)$ in G such that $\mathfrak{S} \cap X$ is a Sylow system of $L \cap X$.

Theorem 4.4.2 in conjunction with Theorem 3.1.5 yields

4.4.4 THEOREM. If M is a maximal subgroup of G f-critical for L such that the Sylow system $\mathfrak{S}$ of L reduces into $L \cap M$, then the f - normalizer of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$ is the f - normalizer of L relative to G corresponding to $\mathfrak{S}$.

It then follows from the fact that relative f - normalizers form a conjugacy class that every relative f - normalizer of L is a relative f - normalizer of some maximal subgroup of G f-critical for L .

We now reimpose the condition that f is integrated.

4.4.5 THEOREM. Let $G_r < G_{r-1} < \ldots < G_0 = G$ be a relative f-critical maximal chain of G (in the sense of Definition 3.4.1) such that $\mathfrak{S} \cap G_{i-1}$ reduces into $L \cap G_i$, $i = 1, 2, \ldots, r$, and suppose that G_r is a minimal member of the chain in the sense that G_r contains no maximal subgroups f-critical for $L \cap G_r$; then G_r is the f - normalizer of L relative to G corresponding to $\mathfrak{S}$. Every relative f - normalizer of L is the minimal member of such a chain.

Proof. By repeated application of (4.4.4) we have that the f - normalizer D of $L \cap G_r$ relative to G_r corresponding to $\mathfrak{G} \cap G_r$ is the f - normalizer of L corresponding to $\mathfrak{G}$. But G_r contains no maximal subgroups f-critical for $L \cap G_r$, and therefore by (4.3.3) $L \cap G_r \in \mathfrak{F}$. It now follows from (4.2.2) that $D = G_r$ as required. That all relative f - normalizers of L may be so expressed follows at once from their conjugacy. We have also proved

4.4.6 THEOREM. If D is an f - normalizer of L relative to G, then $L \cap D \in \mathfrak{F}$.

The chain characterization of relative f - normalizers in (4.4.5) enables us repeatedly to apply specializations of (4.3.6) with N taken as G and L in turn, and with the help of (4.2.2) to deduce

4.4.7 THEOREM. If D is an f-normalizer of L relative to G and if H/K is a chief factor of G f-central for L, then

(a) $\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_D(H \cap D / K \cap D)$, and

(b) $\mathrm{Aut}_L(H/K) \cong \mathrm{Aut}_{L \cap D}(H \cap D / K \cap D)$.

Part (a) of this theorem in conjunction with the covering and avoidance property of relative f - normalizers yields

4.4.8 COROLLARY. The intersection of a relative f - normalizer D of L with a chief series of G is a chief series of D.

We conclude this section with another characterization of relative f - normalizers by maximal chains; the corresponding result for absolute system normalizers is contained in Theorem 4.8 of [13].

Proof. By repeated application of (4.4.4) we have that the f - normalizer D of $L \cap G_r$ relative to G_r corresponding to $\mathfrak{G} \cap G_r$ is the f - normalizer of L corresponding to $\mathfrak{G}$. But G_r contains no maximal subgroups f-critical for $L \cap G_r$, and therefore by (4.3.3) $L \cap G_r \in \mathfrak{F}$. It now follows from (4.2.2) that $D = G_r$ as required. That all relative f - normalizers of L may be so expressed follows at once from their conjugacy. We have also proved

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We conclude this section with another characterization of relative f - normalizers by maximal chains; the corresponding result for absolute system normalizers is contained in Theorem 4.8 of [13].

4.4.9 THEOREM. Let $\mathcal{D}$ be the set of subgroups of G which can be joined to G by a relative f-abnormal maximal chain. Then the minimal members of $\mathcal{D}$ are precisely the f - normalizers of L relative to G.

Proof. Repeated application of (4.4.1) shows that a minimal member D^* of $\mathcal{D}$ contains an f - normalizer D of L relative to G. But by (4.4.5) $D \in \mathcal{D}$, and therefore by minimality $D^* = D$ as required. Since all such D are conjugate, the theorem now follows.

4.5 The previous two sections have been devoted mainly to an investigation of results which are true when the formation function f is integrated. In this section we give examples which show many of the results to be false when this restriction is removed. But we also show that 'subgroup closure' ($f = Sf$) is in some cases an adequate alternative hypothesis to ensure their validity.

4.5.1 EXAMPLE. In the standard notation of this chapter take $L = G = \Sigma_4$, and $f(2) = \{Q, R_o, S\}\Sigma_3$, $f(p) = 1$ for all $p \neq 2$. The function f is clearly not integrated since Σ_3 has an eccentric 3-chief factor. Now G is equal to $G/\emptyset(G) \cap L$ and has a unique minimal normal subgroup E_4 which is an elementary Abelian group of order 4. $E_4/1$ is an f-central chief factor but $G \notin \mathfrak{F}$, and therefore (4.3.2) is not true for a general function f, even when $f = Sf$. Moreover, G has no f-critical maximal subgroup, and the sufficiency of the condition in (4.3.3) does not in general hold. Finally we observe that the 3-chief factor of G is f-critical so that the condition of (4.3.5) is also not in general sufficient.

<u>4.5.2</u> <u>THEOREM</u>. Theorem 4.4.1 remains true when the condition " f is integrated " is relaxed and replaced by the condition " $f = Sf$ ".

<u>Proof</u>. In the notation of (4.4.1) it is again sufficient to show that for $q \neq p$ we have $M \cap N_G(T^q) \leq N_M(\bar{T}^q)$. Now $L/C_q(L) \in f(q)$, and therefore $\bar{L}/C_q(L) \cap M \cong \bar{L}\, C_q(L)/C_q(L) \in Sf(q) = f(q)$. The argument now continues exactly as it does from that point where this relation is established in the proof of (4.4.1).

<u>4.5.3</u> <u>EXAMPLE</u>. Let $W \cong C_5 \wr \Sigma_4$ where Σ_4 is taken with its standard representation of degree 4 in the wreath product. In our standard notation let $L = G \cong W/Z(W)$, and $f(2) = \{Q, R_0, Sn\}\Sigma_3$, $f(5) = \{Q, R_0, Sn\}\Sigma_4$ and $f(p) = 1$ otherwise. $Z(W)$ is the diagonal subgroup of the base group of the wreath product, and G has a unique minimal normal subgroup N of order 5^3 such that G is the splitting extension of N by $L \cong \Sigma_4$; this group has been analysed in detail by Carter in [5]. The formation $f(2)$ comprises groups which may be expressed as the direct product of an elementary Abelian 3-group with a group belonging to the smallest formation containing Σ_3 (already described in (3.4.5)). We now describe $f(5)$: $X \in f(5)$ if and only if X has a normal series $1 \leq K \leq H \leq X$ such that K and X/H are elementary Abelian 2-groups and H/K is an elementary Abelian 3-group, and such that every chief factor R/S below K satisfies $Aut_X(R/S) \cong \Sigma_3$ or C_3 and every chief factor U/V between K and H satisfies $Aut_X(U/V) \cong C_2$ or 1 . Now, returning to G , we see that the 3-chief factor of G is the only f-eccentric one. The f - normalizers of G are easily seen to be of the form $D = NG_2$

where G_2 is a Sylow 2-subgroup of G and is therefore isomorphic with the dihedral group of order 8. Let R/S be a 5-chief factor of D below N which is not centralized by the unique minimal normal subgroup $Z(G_2)$ of G_2 ; such a chief factor exists for if $Z(G_2)$ centralized every 5-chief factor of D below N by a well-known result $Z(G_2)$ would centralize the whole of N which is not the case. Thus $\mathrm{Aut}_D(R/S) \cong G_2$, R/S is an f-eccentric 5-chief factor of D and therefore $D \notin \mathfrak{F}$. Hence an f - normalizer of D is properly contained in D . Since $D < G$, this shows that (4.4.1) is false even when $f = \mathrm{Sn}\, f$.

To see how Theorem 4.4.5 fares when the restriction " f is integrated " is lifted we return for a moment to Example 4.5.1. The group G of that example has no f-critical maximal subgroups and is therefore a minimal member of an f-critical maximal chain. However, its f - normalizer is a proper subgroup so that (4.4.5) is false for non-integrated f even when $f = Sf$. In contrast Theorem 4.4.6 is rescued by S-closure.

<u>4.5.4</u> <u>THEOREM</u>. If $f = Sf$ and D is an f - normalizer of L relative to G , then $L \cap D \in \mathfrak{F}$.

<u>Proof</u>. Let D be the normalizer in G of the f-system $\mathfrak{T}$ of L . The $D = \bigcap_{p \mid |L|} N_p$ where $N_p = N_G(T^p)$ for $f(p)$ - complement $T^p \in \mathfrak{T}$. By (3.3.6) $L \cap N_p \in \mathfrak{F}_p$, and therefore $L \cap D = \bigcap_{p \mid |L|} (L \cap N_p) \in \bigcap_p S\mathfrak{F}_p = \bigcap_p \mathfrak{F}_p = \mathfrak{F}$ as required

Theorem 4.4.6 is not true in general, however. For the group G

in Example 4.5.3 has an absolute f - normalizer D which does not belong to the formation $\mathfrak{F}$. However, in that example the formation $\mathfrak{F}$ may also be defined locally by the integrated formation function f^* specified by taking $f^*(2)$ = the class of elementary Abelian 3-groups, $f^*(5) = \{Q, R_0, Sn\} A_4$ and $f^*(p) = 1$ otherwise. In this case an f^*- normalizer D^* of G has order 2 and is therefore not an f - normalizer of G . Hence f - normalizers are certainly dependent on the way we choose f to define $\mathfrak{F}$, and not simply on $\mathfrak{F}$. In fact our next example shows that for various formation functions f = Sf defining the same $\mathfrak{F}$ locally, the f - normalizers may be widely different from each other, even when each belongs to $\mathfrak{F}$.

<u>4.5.5</u> <u>EXAMPLE</u>. Let $W \cong (C_5 \times C_7) \wr \Sigma_4$ where again the representation of Σ_4 of degree 4 is taken for the wreath product. Set $G \cong W/Z(W)$, and take $f_1(5)$ = the class of 5'-groups with p-length one for all primes p , $f_2(7)$ = the class of 7'-groups with p-length one for all primes p and $f_i(p)$ = the class of p'-groups otherwise, i = 1, 2 . Then f_1 and f_2 are S-closed formation functions defining locally the formation $\mathfrak{F}$ of groups with p-length one for all primes p. For i = 1, 2, let D_i be an absolute f_i- normalizer of G ; then by (4.5.4) $D_i \in \mathfrak{F}$. However, $|D_1| = 2.3.7^3$, $|D_2| = 2.3.5^3$ and therefore D_1 neither contains nor is contained in a conjugate of D_2. If N is the minimal normal subgroup of G of order 7^3 and T is a subgroup of G isomorphic with Σ_3, then

$D_1 = NT$ is an f_1- normalizer of G . However, N decomposes into the direct product of two minimal normal subgroups of D_1 , and therefore (4.4.7) and (4.4.8) no longer hold for general f .

In a similar way one can construct groups containing arbitrarily many distinct conjugacy classes of candidates for the rôle of the relative normalizers associated with a given local formation $\mathfrak{F}$. Fortunately though there is only one class of candidates if we restrict ourselves to integrated formation functions; for an unpublished result due to R.W. Carter and S.E. Stonehewer shows that if f_1 and f_2 are integrated formation functions defining the same $\mathfrak{F}$ locally, then $\mathfrak{R}_p\, f_1(p) = \mathfrak{R}_p\, f_2(p)$ for all primes p. Since the automorphism group of an elementary Abelian p-group has no non-trivial normal p-subgroups, this result shows that the f-centrality of a chief factor of G relative to L depends only on $\mathfrak{F}$ and not on the particular integrated f defining $\mathfrak{F}$. Hence the $f(p)$ - centralizers, and therefore the f-systems of L and their normalizers are the same for every integrated f defining $\mathfrak{F}$. Hence as no ambiguity arises we can make the following definition.

<u>4.5.6</u> <u>DEFINITION</u>. If $\mathfrak{F}$ is an arbitrary saturated formation we define the <u>$\mathfrak{F}$- normalizers</u> of L relative to G to be the f - normalizers of L relative to G for any integrated formation function f defining $\mathfrak{F}$ locally, for as remarked in (2.2.1) we know that at least one such f always exists.

Our next result shows that $\mathfrak{F}$- normalizers are always contained in corresponding f - normalizers.

<u>4.5.7</u> <u>THEOREM</u>. If f and f^* are formation functions defining the same local formation $\mathfrak{F}$, and if f is integrated, then an f^*- normalizer of L relative to G contains an f - normalizer of L relative to G.

<u>Proof</u>. We use induction on $|L|$. Let D be an f - normalizer of L relative to G. By (4.2.2) if $L \in \mathfrak{F}$ the f - and f^*- normalizers of L relative to G coincide with G, and the result is true. Thus we may assume that $L \notin \mathfrak{F}$ and therefore by (4.3.3) and (4.4.4) that G has a maximal subgroup M f-critical for L such that D is an f - normalizer of $L \cap M$ relative to M. Since $|L \cap M| < |L|$, by induction D is contained in an f^*- normalizer $\bar{D}$ of $L \cap M$ relative to M. Theorem 4.4.2 is true for an arbitrary formation function and therefore, since $M\,F(L) = G$, we have $\bar{D} = M \cap D^*$ for a suitable f^*- normalizer D^* of L relative to G. Thus $D \leq D^*$ as required.

This theorem provides us with a proof of the fact that f - normalizers depend only on $\mathfrak{F}$ when f is integrated, a proof which is independent of the theorem of Carter and Stonehewer cited above; for if f and f^* are both integrated in (4.5.7) we have $D \leq D^*$, $D^* \leq D$, and therefore equality. In order to illustrate this situation we return to Example 4.5.5. There the function f specified by taking $f(p)$ to be the class of p'-groups with q-length one for all primes q, for each p, is an integrated function defining $\mathfrak{F}$. The subgroup T of order 6 is an $\mathfrak{F}$- normalizer of G and is properly contained in the f_i- normalizer D_i, $i = 1, 2$.

Let $\mathfrak{D}$ be the set of subgroups of G described in the statement of (4.4.9). Although we suspect that for a general formation function f it is not true that a relative f - normalizer belongs to $\mathfrak{D}$, even in the case $f = Sf$, we have been unable to find a counterexample. Nevertheless (4.4.9) is certainly false in general; for as we pointed out the group G in Example 4.5.3 has an f - normalizer D which does not belong to $\mathfrak{F}$, and therefore D contains f-abnormal maximal subgroups. But D itself is an f-abnormal maximal subgroup of G, and therefore cannot be a minimal member of the appropriate set $\mathfrak{D}$ of subgroups. When $f = Sf$, Theorem 4.5.2 shows that minimal members of $\mathfrak{D}$ certainly contain relative f - normalizers. Therefore if our suspicion is wrong and relative f - normalizers do belong to the set $\mathfrak{D}$ when $f = Sf$, then (4.4.9) will also be true in this case.

We conclude this section with a theorem which is true under the two hypotheses (a) f is integrated, and (b) $f = Sn\, f$, but which becomes false if either is omitted. It relates different relative f - normalizers in the case when G (as well as L) is soluble.

<u>4.5.8</u> <u>THEOREM</u>. Suppose that G is soluble and that $f = Sn\, f$ is an integrated formation function defining the formation $\mathfrak{F}$ locally. Let D be an f - normalizer of L relative to G, and let $L \leq K \triangleleft G$. If N is an f - normalizer of $K \cap D$ relative to D, then N is an f - normalizer of K relative to G.

Proof. We use induction on $|G|$. If $D = G$ the result is trivially true. Therefore assume $D < G$, and by (4.4.5) let M be a maximal subgroup of G belonging to a relative f-critical maximal chain from D up to G; then D is an f - normalizer of $L \cap M$ relative to M. Since $L \cap M \leq K \cap M \lhd M$ and $|M| < |G|$, by induction an f - normalizer of $(K \cap M) \cap D$ relative to D is an f - normalizer of $K \cap M$ relative to M. But $(K \cap M) \cap D = K \cap D$, and therefore N is an f - normalizer of $K \cap M$ relative to M. It is now sufficient to show that M is a maximal subgroup of G f-critical for K, for then by (4.4.4) N is an f - normalizer of K relative to G as required. Let R/S be a chief factor of G complemented by M. Let p be the prime dividing $|R:S|$ and write $C = C_K(R/S)$. Since M is f-critical for L, by (2.3.6) R/S is f-eccentric for L, and we have $f(p) \not\ni L/C_L(R/S) = L/L \cap C \cong LC/C \lhd K/C$. Hence, since $f(p)$ is closed under taking normal subgroups, we must have $K/C \notin f(p)$. This means R/S is f-eccentric for K and again by (2.3.6) M is f-abnormal for K. Thus condition (i) of (4.3.1) is satisfied by M. But condition (ii) follows at once from the fact that $F(L) \leq F(K)$, and therefore M is f-critical for K as required to complete the proof.

If we take $K = G$ in (4.5.8) we see that an absolute f - normalizer of an f - normalizer of L relative to G is an absolute f - normalizer of G. In (4.5.3) we have an example of a group G and a formation function $f = \text{Sn}\, f$ such that the f - normalizers of G do not belong to $\mathfrak{F}$. However, if we specialize (4.5.8) still further by taking $L = G$, it states that an f - normalizer of an f - normalizer of G is an f - normalizer of G ; in other words, the f - normalizers of G belong to $\mathfrak{F}$. Hence (4.5.8) is false if hypothesis (a) is omitted. On the other hand, if we take $K = G \cong \Sigma_4$ and $L \cong A_4$ with $f(2) = \{Q, R_o\} \Sigma_3$ and $f(p) = \mathfrak{F}_2$ for $p \neq 2$, the function f is integrated but not Sn-closed. As shown in (3.4.5) an $\mathfrak{F}$- normalizer of L relative to G has order 6 whereas $G \in \mathfrak{F}$ and is its own $\mathfrak{F}$- normalizer. This shows (4.5.8) is also false when hypothesis (b) is omitted.

<u>4.6</u> We now take $L = G$ and conclude this chapter with a summary of our main results as they apply to the properties of an absolute f - normalizer of a soluble group.

$C_p(G)$ is the intersection of the centralizers of the f-central p-chief factors of G , and if S^p is a Sylow p-complement of G belonging to the Sylow system $\mathfrak{S}$, then $T^p = S^p \cap C_p(G)$ is the $f(p)$ - complement of G corresponding to $\mathfrak{S}$. The complete set $\{T^p\}$ of $f(p)$ - complements generates the f-system $\mathfrak{J}$ of G corresponding to $\mathfrak{S}$. $N_G(T^p)$ is an $f(p)$ - normalizer of G and the intersection of a complete set of these is the normalizer of $\mathfrak{J}$, namely the absolute f - normalizer of G corresponding to $\mathfrak{S}$. The set of all f - normalizers of G forms a homomorphism-invariant characteristic conjugacy class whose intersection is the f-hypercentre of G . An f - normalizer D of G covers the f-central chief factors and avoids the f-eccentric chief factors of G , and therefore $|G:D|$, the number of f-systems of G , is equal to the product of the orders of the f-eccentric chief factors in a given chief series of G .

Now suppose that f is integrated. A maximal subgroup of G is f-critical if and only if it complements an f-critical chief factor of G , and G has f-critical maximal subgroups if and only if G does not belong to the formation $\mathfrak{F}$. If M is an f-abnormal maximal subgroup of G into which $\mathfrak{G}$ reduces, then the f - normalizer D of G corresponding to $\mathfrak{G}$ is contained in the f - normalizer of M corresponding to $\mathfrak{G} \cap M$; this is also true when f is not necessarily integrated providing $f = Sf$. Further, if M is any maximal subgroup of G supplementing $F(G)$, then for general f an f - normalizer of M is the intersection of M with a suitable f - normalizer of G ; therefore an f - normalizer of an f-critical maximal subgroup is also an f - normalizer of G . For integrated f the f - normalizers of G are precisely the minimal members of the f-critical maximal chains of G , and they may also be characterized as the minimal members of the set $\mathfrak{D}$ comprising all those subgroups of G which may be joined to G by an f-abnormal maximal chain. In general f - normalizers are not $\mathfrak{F}$-groups, but if the defining formation function f is either integrated or S-closed they do in fact belong to the class $\mathfrak{F}$. In the former case the f - normalizers depend only on $\mathfrak{F}$ and are therefore called the $\mathfrak{F}$- normalizers of G; for any f defining $\mathfrak{F}$ locally an f - normalizer always contains an $\mathfrak{F}$ - normalizer.

4.6.1 THEOREM. If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are local formations such that $\mathfrak{F}_2 \leq \mathfrak{F}_1$, then an $\mathfrak{F}_2$- normalizer D_2 of an $\mathfrak{F}_1$- normalizer D_1 of G is an $\mathfrak{F}_2$- normalizer of G.

Proof. Suppose $\mathfrak{F}_i$ is defined locally by the integrated formation function f_i, $i = 1,2$. Write $f_2^*(p) = f_1(p) \cap f_2(p)$ for all primes p; then f_2^* is a formation function defining a local formation, $\mathfrak{F}_2^*$ say, and clearly $\mathfrak{F}_2^* \leq \mathfrak{F}_2$. Let $G \in \mathfrak{F}_2$; then $X = G/O_{p'p}(G) \in f_2(p)$. But since $\mathfrak{F}_2 \leq \mathfrak{F}_1$, we have $G \in \mathfrak{F}_1$, $X \in f_1(p)$ and therefore $X \in f_2^*(p)$. Hence $G \in \mathfrak{F}_2^*$ and so $\mathfrak{F}_2 = \mathfrak{F}_2^*$. Therefore without loss of generality we may assume $f_2(p) \leq f_1(p)$ for all primes p. If D is an $\mathfrak{F}_1$- normalizer of G, by (4.4.5) G has an f -critical maximal chain $D = G_r < G_{r-1} < \ldots < G_0 = G$. Now G_i is an f_1-abnormal maximal subgroup of G_{i-1}, and is therefore f_2-abnormal in G_{i-1}, $i = 1, 2, \ldots, r$. Hence we have an f_2-critical maximal chain from D_1 up to G. Again by (4.4.5) D_2 may be joined to D_1 (and hence to G) by an f_2-critical maximal chain, and is therefore an $\mathfrak{F}_2$- normalizer of G as claimed.

When $f(p) = 1$ for all primes p the f - normalizers are simply the system normalizers of G. Since by definition of a formation $1 \leq f(p)$, Theorem 4.6.1 shows that an $\mathfrak{F}$- normalizer - and hence by (4.5.7) an f - normalizer - always contains a system normalizer of G, for any formation function defining $\mathfrak{F}$ locally. P. Hall showed in [13] that the join of the system normalizers of a group is the whole group. From these facts and from (4.2.5) follows

4.6.2 THEOREM. If L is a soluble normal subgroup of an arbitrary group G, and if f is an arbitrary formation function, then the join of the f - normalizers of L relative to G is G itself.

Finally, with (4.6.1) in mind, we outline the proof of a theorem about the maximal chains of a soluble group.

4.6.3 THEOREM. A soluble group G has a maximal chain of the form

$$1 = G_0 < G_1 < \dots < G_r < G_{r+1} < \dots < G_s < \dots < G_t = G ,$$

where (i) $G_{i-1} \triangleleft G_i$ for $i = 1, \dots, r$,

(ii) G_{i-1} is non-normal of prime index in G_i for $i = r+1, \dots s$,

and (iii) $|G_i : G_{i-1}|$ is not a prime for $i = s+1, \dots, t$.

This chain may be chosen so that in a given chief series of G to each link $G_{i-1} < G_i$ there corresponds a chief factor H/K of G which is covered by G_i and avoided by G_{i-1}. Thus t is equal to the number of chief factors in a chief series of G, and to each index $|G_i : G_{i-1}|$ there corresponds a chief factor in the given chief series of the same order.

Proof. Let f_i define $\mathfrak{F}_i$ locally, $i = 1, 2$, where $f_1(p)$ is the class of $\mathfrak{A}$-groups of exponent $p - 1$ and $f(p) = 1$ for all primes p. Then by (2.2.7) $\mathfrak{F}_1 = \mathfrak{U}$ and $\mathfrak{F}_2 = \mathfrak{N}$, and clearly f_i is integrated, $i = 1, 2$. By (4.6.1) we may choose the above chain with G_s a $\mathfrak{U}$- normalizer of G joined to G by an f_1-critical maximal chain, and G_r a system normalizer of G joined to G_s by an f_2-critical chain. Since the $\mathfrak{U}$- normalizers of a group cover the cyclic chief factors and avoid the rest, the indices above G_s are not primes; since an f_2-critical maximal subgroup of a supersoluble group complements an eccentric cyclic chief factor, the links between G_r and G_s are non-normal of prime index; and since maximal subgroups of $\mathfrak{N}$-groups are normal, so therefore are the links below G_r. The rest of the theorem is clear in view of the properties of maximal subgroups supplementing the Fitting subgroup as described in (4.3.6), (4.3.7) and (4.3.8).

In [16], Theorem 1, Huppert shows that the p-chief rank of a soluble group is equal to the highest power of p occuring as an index in any maximal chain of G. Theorem 4.6.3 sheds more light on this result by showing that all the orders of chief factors of G occur as indices in a single maximal chain of G.

Chapter Five

RELATIVE $\mathfrak{F}$- COVERING SUBGROUPS

5.1 This chapter contains two new characterizations of Gaschütz's $\mathfrak{F}$ - covering subgroups first introduced and investigated by him in [8]. In addition we extend their definition in a familiar direction by constructing $\mathfrak{F}$ - covering subgroups of a soluble group L relative to an arbitrary group G which has L as a normal subgroup. In contrast to the situation for f - normalizers in chapters three and four, here we investigate the general theory of $\mathfrak{F}$ - covering subgroups first before looking at the p-theory which appears later as a special case. Although in [8] Gaschütz proves the existence of $\mathfrak{F}$ - covering subgroups and many of their important properties, it is not clear at first sight how to construct them for a given group. The first of our two characterizations does in fact yield a construction for $\mathfrak{F}$ - covering subgroups, and we feel this approach is sufficiently different and we hope of sufficient interest to justify our developing the main properties of the subgroups afresh using this characterization as a definition. Our account is therefore self-contained, and our proofs are independent of the properties $\mathfrak{F}$ - covering subgroups were shown to possess in [8]. We do not need the fact that f is integrated until section 5.3, and therefore in the first two sections of this chapter f will denote an arbitrary formation function defining $\mathfrak{F}$ locally.

<u>5.1.1</u> <u>DEFINITIONS</u>. A maximal subgroup M of G is called <u>$\mathfrak{F}$ -crucial for L</u> if

(i) M is f-abnormal for L, and

(ii) $L \cap M / L \cap \mathrm{Core}(M) \in \mathfrak{F}$.

A chief factor H/K of G is called <u>$\mathfrak{F}$-crucial for L</u> if

(iii) H/K is f-eccentric for L, and

(iv) $LH/H \in \mathfrak{F}$.

<u>5.1.2</u> <u>LEMMA</u>. Conditions (iii) and (iv) of (5.1.1) are equivalent to

(iii)' $L/L \cap K \notin \mathfrak{F}$, and

(iv)' $L/L \cap H \in \mathfrak{F}$.

<u>Proof</u>. Conditions (iv) and (iv)' are equivalent by the standard isomorphism $LH/H \cong L/L \cap H$. Suppose then that (iii) and (iv) hold for H/K. If L avoids H/K, then $L \cap H = L \cap K$ and L centralizes H/K. But this possibility is ruled out by (iii); therefore L covers H/K and $L \cap H/L \cap K \underset{G}{\cong} (L \cap H)K/K = H/K$. Thus $L \cap H/L \cap K$ is a chief factor of G f-eccentric for L. Hence by (2.4.7) we have $L/L \cap K \notin \mathfrak{F}$. Conversely suppose conditions (iii)' and (iv)' hold for H/K. They imply that $L \cap H/L \cap K$ is non-trivial and therefore that L covers H/K. As above $L \cap H/L \cap K \underset{G}{\cong} H/K$, and $L \cap H/L \cap K$ is a chief factor of G which by (2.4.7) must be f-eccentric for L. Therefore H/K is also f-eccentric for L and (iii) holds. This completes the proof.

<u>5.1.3</u> <u>THEOREM</u>. A maximal subgroup M of G $\mathfrak{F}$-crucial for L complements a chief factor of G $\mathfrak{F}$-crucial for L. Conversely,

a chief factor H/K of G $\mathfrak{F}$-crucial for L is always complemented by a maximal subgroup of G $\mathfrak{F}$-crucial for L.

Proof. Suppose M is $\mathfrak{F}$-crucial for L. If M contained L it would be f-normal contradicting condition (i) of (5.1.1); therefore $ML = G$. By (2.3.7) $L/L \cap \mathrm{Core}(M)$ has a subgroup $V/L \cap \mathrm{Core}(M)$ which is a chief factor of G and is complemented by M. Moreover $LV/V = L/V \cong L \cap M/L \cap \mathrm{Core}(M) \in \mathfrak{F}$. But M is f-abnormal for L and therefore by (2.3.6) $V/L \cap \mathrm{Core}(M)$ is f-eccentric for L. Hence $V/L \cap \mathrm{Core}(M)$ is $\mathfrak{F}$-crucial for L. Conversely, suppose H/K is a chief factor of G $\mathfrak{F}$-crucial for L. Since $LH/H \in \mathfrak{F}$ and $LH/K \notin \mathfrak{F}$, it follows from (5.1.6) below that H/K is complemented in G, by M say. Since H/K is f-eccentric for L, by (2.3.6) M is f-abnormal for L and condition (i) of (5.1.1) is satisfied by M. It therefore remains to confirm that $L \cap M/L \cap \mathrm{Core}(M) \in \mathfrak{F}$. As shown in (5.1.2) $L \cap H/L \cap K$ is a chief factor of G; moreover, since $H \cap M = K$, we have $(L \cap H) \cap M = L \cap K$, and therefore M complements $L \cap H/L \cap K$. Hence $L/L \cap H = (L \cap M)(L \cap H)/L \cap H \cong L \cap M/L \cap K$, and therefore $L \cap M/L \cap \mathrm{Core}(M) \in Q(L \cap M/L \cap K) = Q(L/L \cap H) \leq Q\mathfrak{F} = \mathfrak{F}$, as required.

5.1.4 THEOREM. Let R denote the $\mathfrak{F}$-residual of L. Then a maximal subgroup M of G is $\mathfrak{F}$-crucial for L if and only if $R/R \cap M$ is a chief factor of G.

Proof. We first observe that $R \trianglelefteq L \triangleleft G$ and therefore $R \triangleleft G$. Now if $R/R \cap M$ is a chief factor of G it must be f-eccentric for

for L, for otherwise by (2.4.7) we should have $L/R \cap M \in \mathfrak{F}$ contradicting the definition of R. Hence conditions (iii) and (iv) of (5.1.1) are satisfied and $R/R \cap M$ is $\mathfrak{F}$-crucial for L. But M complements $R/R \cap M$ and therefore by (5.1.3) M is also $\mathfrak{F}$-crucial for L. Conversely suppose M is $\mathfrak{F}$-crucial for L. Let $W = L \cap \mathrm{Core}(M)$ and let V/W be the unique minimal normal subgroup of G/W contained in L/W - compare (2.3.7). Now $L/V = L \cap M/W \in \mathfrak{F}$ by hypothesis, and therefore $R \leq V$. Since V/W is f-eccentric for L by (2.3.6), we have $L/W \notin \mathfrak{F}$ by (2.4.7); hence $R \nleq W$ and therefore $RW = V$. Thus $V/W \underset{G}{\cong} R/R \cap W$ and it follows that $R/R \cap W$ is a chief factor of G f-eccentric for L. If $R \leq M$, then M complements some chief factor of G between R and L; but by (2.3.6) this is impossible since all these chief factors are f-central for L whereas M is f-abnormal for L. Hence $R \nleq M$. Since M contains W and therefore a fortiori $R \cap W$, it complements the chief factor $R/R \cap W$. Therefore $R \cap M = R \cap W$, and the result follows.

<u>5.1.5</u> <u>THEOREM</u>. G has maximal subgroups $\mathfrak{F}$-crucial for L if and only if $L \notin \mathfrak{F}$.

<u>Proof</u>. The necessity is an immediate consequence of (5.1.4). The sufficiency follows from (5.1.3); for if $L \notin \mathfrak{F}$, the $\mathfrak{F}$-residual of L is non-trivial and so G has $\mathfrak{F}$-crucial chief factors.

In order to prove the first main theorem of this chapter we need to generalize slightly two well-known results.

<u>5.1.6</u> <u>LEMMA</u>. Let M_1 and M_2 be conjugate maximal subgroups of G

abnormal for L (that is p-maximal f-abnormal for L with $f(p) = 1$). If both contain the p-complement L^p of L then $M_1 = M_2$.

Proof. Putting $f(p) = 1$ in (3.1.5) we have $N_G(L^p) \leq M_i$, and the result now follows at once from Theorem 3.7 of [13].

The next lemma ties in with Gaschütz's definition of a saturated formation (see (2.3) of [8]) and the subsequent amendment (see Theorem 1 of [9]).

5.1.7 LEMMA. Let N be a minimal normal subgroup of G contained in L such that $L/N \in \mathfrak{F}$ and $L \notin \mathfrak{F}$. Then N is complemented and all complements are conjugate.

Proof. As the arguments are well-known we give only an outline. By (2.4.7) $N \not\leq \emptyset(G)$, and therefore N is complemented in G, by M say. Write $W = L \cap \mathrm{Core}(M)$. If $W \neq 1$, G has a minimal normal subgroup $N^* \leq W$, and by induction on $|L|$ the result is true for G/N^*. Clearly $N^* \neq N$ so that N^*N/N is a minimal normal subgroup of G/N which is f-central for L/N, and therefore by the standard G-isomorphism $N^*/1$ is f-central for L. By (2.3.6) complements of N in G contain N^*; they therefore complement NN^*/N^* in G/N^* and by induction are all conjugate. On the other hand if $W = 1$ we have $N = C_L(N)$. Let p be the prime dividing $|N|$. Since $L/N \neq 1$ a minimal normal subgroup K/N of G/N contained in L/N has order a power of $q \neq p$. By the Frattini argument the complements of N in G coincide with the normalizers of the Sylow q-subgroups of K, and it follows from Sylow's Theorem that these are all conjugate.

We now come to the main theorem of this section; it leads directly to our constructive definition of relative $\mathfrak{F}$- covering subgroups.

5.1.8 THEOREM. Let $\mathfrak{S}$ be a Sylow system of L and let E be a minimal member of a chain of the form

$$E = E_r < E_{r-1} < \dots < E_1 < E_0 = G \qquad (*)$$

where E_i is a maximal subgroup of E_{i-1} $\mathfrak{F}$-crucial for $L \cap E_{i-1}$ and is such that $\mathfrak{S} \cap E_{i-1}$ reduces into $L \cap E_i$, $i = 1, 2, \dots, r$. Then E is uniquely determined and $L \cap E \in \mathfrak{F}$.

Proof. We first prove the uniqueness of E using induction on $|L|$. If $L \in \mathfrak{F}$ the result is certainly true, for then by (5.1.5) G has no maximal subgroups $\mathfrak{F}$-crucial for L and G is itself the unique minimal member of the chain (*). Therefore we may assume $L \notin \mathfrak{F}$. In this case L has a non-trivial $\mathfrak{F}$-residual , R say, and G has maximal subgroups $\mathfrak{F}$-crucial for L so that $r \geq 1$. Suppose E^* is the minimal member of another relative $\mathfrak{F}$-crucial chain into which $\mathfrak{S}$ reduces, viz. $E^* = E^*_s < E^*_{s-1} < \dots < E^*_1 < E^*_0 = G$, $s \geq 1$. Write $T = R \cap E_1$ and $T^* = R \cap E^*_1$. By (5.1.4) R/T and R/T^* are chief factors of G f-eccentric for L . If $T = T^*$, then by (5.1.6) and (5.1.7) we have $E_1 = E^*_1$. Since $|E_1 \cap L| < |L|$, the result now follows by induction. Therefore suppose $T \neq T^*$. We show that in this case $E_1 \cap E^*_1$ is a maximal subgroup of E_1 $\mathfrak{F}$-crucial for $L \cap E_1$. $T \neq T^*$ implies that $TT^* = R$, and therefore $R/T^* \underset{G}{\cong} T/T \cap T^*$. Moreover $G/R \underset{G}{\cong} E_1/T$, and therefore $T/T \cap T^*$, as a chief factor of E_1 f-eccentric for $L \cap E_1$, is $\mathfrak{F}$-crucial for $L \cap E_1$. Since $T \nleq E^*_1$, we have $T(E_1 \cap E^*_1) = E_1 \cap TE^*_1 = E_1$ and therefore

$T/T \cap T^*$ is complemented by $E_1 \cap E_1^*$ in E_1 . Hence by (5.1.3) $E_1 \cap E_1^*$ is a maximal subgroup of E_1 $\mathfrak{F}$-crucial for $L \cap E_1$, and $\mathfrak{S} \cap E_1$ clearly reduces into $E_1 \cap E_1^* \cap L$. Similarly $E_1 \cap E_1^*$ is a maximal subgroup of E_1^* $\mathfrak{F}$-crucial for $L \cap E_1^*$. Hence applying our induction hypothesis to E_1 we see that E is the minimal member of every relative $\mathfrak{F}$-crucial maximal chain of E_1 into which $\mathfrak{S} \cap E_1$ reduces. In particular, E can be joined to $E_1 \cap E_1^*$ (and therefore to E_1^*) by a chain of the corresponding form. Once more applying our induction hypothesis, this time to E_1^* , since both E and E^* are now minimal members of such chains of E_1^* into which $\mathfrak{S} \cap E_1^*$ reduces, we must have $E = E^*$.

It remains to prove that $L \cap E \in \mathfrak{F}$; but this follows at once from (5.1.5) since by definition E has no maximal subgroups $\mathfrak{F}$-crucial for $L \cap E$.

<u>5.2</u> We are now in a position to make the fundamental definition of this chapter.

<u>5.2.1</u> <u>DEFINITION</u>. The uniquely defined subgroup E of Theorem 5.1.8 is called the <u>$\mathfrak{F}$ - covering subgroup of L relative to G corresponding to the Sylow system $\mathfrak{S}$</u> .

It follows at once from the definition and (5.1.5) that G is itself a relative $\mathfrak{F}$- covering subgroup of L if and only if $L \in \mathfrak{F}$. Our next result is an immediate consequence of the fact the Sylow systems of L are permuted transitively by the inner automorphisms of G .

5.2.2 THEOREM. The $\mathfrak{F}$- covering subgroups of L relative to G form a conjugacy class of G, and if $L \triangleleft G$ this conjugacy class is invariant under automorphisms of G.

5.2.3 LEMMA. If E is an $\mathfrak{F}$- covering subgroup of L relative to G, and if R is the $\mathfrak{F}$-residual of L, then $ER = G$.

Proof. We use induction on $|L|$. If $R = 1$ then $L \in \mathfrak{F}$; in this case $E = G$ and the lemma is true. If $R \neq 1$, let M be a maximal subgroup of G in a relative $\mathfrak{F}$-crucial maximal chain from E up to G. E is an $\mathfrak{F}$-covering subgroup of $L \cap M$ relative to M, $|L \cap M| < |L|$, and so by induction $ER^* = M$ where R^* is the $\mathfrak{F}$-residual of $L \cap M$. It follows from (5.1.4) that $RM = G$ and therefore $R(L \cap M) = L$; hence $L \cap M/R \cap M = L/R \in \mathfrak{F}$. Therefore $R^* \leq R \cap M$ and so $ER \geq ER^* = M$. Therefore $ER = ERM = G$ as required.

5.2.4 THEOREM. If E is an $\mathfrak{F}$- covering subgroup of L relative to G, and if $E \leq X \leq G$, then E is an $\mathfrak{F}$- covering subgroup of $L \cap X$ relative to X.

Proof. We use induction on $|L|$. If $L \in \mathfrak{F}$ then $E = G$ and the result is clearly true. Therefore suppose $L \notin \mathfrak{F}$, and let M be a maximal subgroup of G belonging to a relative $\mathfrak{F}$-crucial maximal chain from E up to G. We distinguish two cases:

Case 1. $X \leq M$. Since E is the minimal member of a relative $\mathfrak{F}$-crucial maximal chain of M, E is an $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M. Since $|L \cap M| < |L|$ and $E \leq X \leq M$ we may to conclude that E is an $\mathfrak{F}$- covering subgroup of $L \cap M \cap X$ $= L \cap X$ relative to X.

Case 2. $X \not\leq M$. By (5.2.3) E, and therefore X, covers G/R where R is the $\mathfrak{F}$-residual of L. Moreover, by (5.1.4) $R/R \cap M$ is a chief factor of G. Suppose $X(R \cap M) \neq G$; now $X(R \cap M) \geq E(R \cap M)$, and since E covers G/R, $E(R \cap M)$ is either equal to G or complements $R/R \cap M$. Since $E(R \cap M) \leq M$ the first possibility is ruled out, and therefore $E(R \cap M) = M$. Hence $M \leq X(R \cap M)$ and since M is maximal and $X \not\leq M$, we must have $X(R \cap M) = G$. Thus $G/R \cap M \underset{G}{\cong} X/R \cap M \cap X$ which means that $R \cap X/R \cap M \cap X$ is a chief factor of X $\mathfrak{F}$-crucial for $L \cap X$. Moreover, we have $(X \cap M)(R \cap M) = X(R \cap M) \cap M = M$, and therefore $G = MR = (X \cap M)(R \cap M)R = (X \cap M)R$. Hence $(X \cap M)(R \cap X) = (X \cap M)R \cap X = X$ which shows that $X \cap M$ complements the chief factor $R \cap X/R \cap M \cap X$ in X. Therefore $X \cap M$ is a maximal subgroup of X $\mathfrak{F}$-crucial for $L \cap X$ by (5.1.3). Now E is an $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M and $E \leq X \cap M \leq M$; therefore, since $|L \cap M| < |L|$, by induction E is an $\mathfrak{F}$- covering subgroup of $L \cap X \cap M$ relative to $X \cap M$ and hence can be joined to $X \cap M$ by a relative $\mathfrak{F}$-crucial maximal chain. Thus E is the minimal member of a relative $\mathfrak{F}$-crucial maximal chain of X, and is therefore an $\mathfrak{F}$- covering subgroup of $L \cap X$ relative to X. This completes the proof.

We are now in a strong position to prove a property of relative

$\mathfrak{F}$ - covering subgroups which in the case $L = G$ specializes to the property used by Gaschütz to characterize them in [8].

<u>5.2.5</u> <u>THEOREM</u>. E is an $\mathfrak{F}$ - covering subgroup of L relative to G if and only if E is a subgroup of G satisfying

(a) $LE = G$,

(b) $L \cap E \in \mathfrak{F}$, and

(c) whenever $E \leq X \leq G$ and $X^* \triangleleft L \cap X$ such that $L \cap X/X^* \in \mathfrak{F}$ then $X^*(L \cap E) = L \cap X$.

<u>Proof</u>. We first show the necessity of conditions (a), (b) and (c). Let E be an $\mathfrak{F}$- covering subgroup of L relative to G ; then by (5.2.3) E satisfies (a), and by (5.1.8) it satisfies (b). To prove that condition (c) is satisfied we observe that by (5.2.4) E is an $\mathfrak{F}$- covering subgroup of $L \cap X$ relative to X . If R is the $\mathfrak{F}$-residual of $L \cap X$, then by (5.2.3) $ER = X$ and hence $R(L \cap E) = L \cap X$. If $L \cap X/X^* \in \mathfrak{F}$, then $R \leq X^*$ and therefore $X^*(L \cap E) = L \cap X$ as required.

To prove the sufficiency, suppose conversely that E is a subgroup of G satisfying (a), (b), and (c). Let R be the $\mathfrak{F}$-residual of L . Conditions (a) and (c) imply in turn that E covers G/L and L/R , and therefore $ER = G$. If $R = 1$ then $E = G$ and the theorem is true; for in this case $L \in \mathfrak{F}$ and G is the unique $\mathfrak{F}$ - covering subgroup of L relative to G. Therefore suppose $R \neq 1$ and let R/T be a chief factor of G ; clearly R/T is $\mathfrak{F}$-crucial for L . Now suppose $ET = G$; then $(L \cap E)T = L$, and therefore $L/T \cong L \cap E/T \cap E \in Q(L \cap E) \leq Q\mathfrak{F} = \mathfrak{F}$ by condition (b). But then $R \leq T$ which is impossible. Thus $G \neq ET$, and so writing $ET = M$ we have $MR \geq ER = G$ and $R > R \cap M \geq T$. Hence M complements the chief factor R/T in G, and is therefore by (5.1.3) a maximal subgroup of G $\mathfrak{F}$-crucial for L. By conditions (a), (b) and (c) we therefore have for M

(a)' $(L \cap M)E = M$,

(b)' $(L \cap M) \cap E = L \cap E \in \mathfrak{F}$, and

(c)' whenever $E \leq X \leq M$ and $X^* \triangleleft L \cap X$ such that $L \cap X/X^* \in \mathfrak{F}$ then $X^*(L \cap M \cap X) = L \cap M \cap X$.

Now $LM = LE = G$ by condition (a), and therefore $|L \cap M| < |L|$. Hence by induction E is an $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M, and since M is $\mathfrak{F}$-crucial for L, E is therefore an $\mathfrak{F}$ - covering subgroup of L relative to G as required.

The next theorem shows that relative $\mathfrak{F}$- covering subgroups are homomorphism invariant.

<u>5.2.6</u> <u>THEOREM</u>. If $\theta : G \longrightarrow G^*$ is a homomorphism of G onto G^*, and if E is an $\mathfrak{F}$- covering subgroup of L relative to G then $E^* = \theta(E)$ is an $\mathfrak{F}$ - covering subgroup of $L^* = \theta(L)$ relative to G^*.

<u>Proof</u>. Let N be the kernal of θ; then it is sufficient to show that EN/N is an $\mathfrak{F}$ - covering subgroup of LN/N relative to G/N. We use induction on $|L|$. If $LN/N \in \mathfrak{F}$ then by the remark after

Definition 5.2.1 G/N is an $\mathfrak{F}$- covering subgroup of LN/N relative to G/N ; but $LN/N \cong L/L \cap N$, and therefore if R is the $\mathfrak{F}$-residual of L we have $R \leq L \cap N \leq N$. By (5.2.3) $ER = G$; hence $EN = G$ and the result is true in this case. Therefore we may suppose $LN/N \notin \mathfrak{F}$, so that G has a chief factor H/K between N and LN which is $\mathfrak{F}$-crucial for L. Let M be a complement of H/K in G . By (5.1.3) M is $\mathfrak{F}$-crucial for L and by the conjugacy of relative $\mathfrak{F}$- covering subgroups M may be chosen to contain E ; clearly M/N is a maximal subgroup of G/N $\mathfrak{F}$-crucial for LN/N . Since $|L \cap M| < |L|$ and E is an $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M , by induction EN/N is an $\mathfrak{F}$-covering subgroup of $(L \cap M)N/N$ relative to M/N . But $(L \cap M)N/N = LN/N \cap M/N$ and therefore EN/N can be joined to M/N , and therefore to G/N , by a relative $\mathfrak{F}$-crucial chain; hence EN/N is by definition an $\mathfrak{F}$- covering subgroup of LN/N relative to G/N as required.

From the fact that $\mathfrak{F}$- covering subgroups of L relative to G form a conjugacy class of G we now deduce

<u>5.2.7</u> <u>COROLLARY</u>. If $N \triangleleft G$ and E^*/N is an $\mathfrak{F}$- covering subgroup of LN/N relative to G/N , then $E^* = EN$ for a suitable $\mathfrak{F}$- covering subgroup E of L relative to G .

We recall that up to this point in our work $f(p)$ has been a formation and therefore by definition non-empty so that the local formation $\mathfrak{F}$ defined by f has therefore always contained the class of nilpotent groups. If we now take $f(p) = \emptyset$, the empty set, for some p, and adopt the convention that every p-chief factor of G is f-eccentric for L, then $\mathfrak{F}$ is some class of soluble p'-groups. If we take L to be a minimal normal p-subgroup of G contained in $\Phi(G)$, then $L/1$ is a chief factor of G $\mathfrak{F}$-crucial for L. But $L/1$ is not complemented in G and therefore L has no $\mathfrak{F}$- covering subgroup relative to G. Therefore we cannot allow $f(p)$ to be empty in our theory of relative $\mathfrak{F}$-covering subgroups. However, if we take $L = G$ in sections 5.1 and 5.2, it is not difficult to verify that the proofs of the results in these sections go through with the assumption $f(p) = \emptyset$ for one or more primes p. For example if $G/R \in \mathfrak{F}$ and R/T is a p-chief factor, then with $f(p) = \emptyset$ it is clear that R/T is an $\mathfrak{F}$-crucial chief factor of G. Since G/R is now a soluble p'-group, G/T has a Sylow p-complement M/T and M is the unique complement of R/T into which a given Sylow system $\mathfrak{S}$ of G reduces. This is the only additional fact needed to extend the proof of the basic Theorem 5.1.8 to this more general situation. Therefore in the theory of absolute $\mathfrak{F}$ - covering subgroups we may allow the possibility $f(p) = \emptyset$ as did Gaschütz in [8].

5.3 The definitions and results of the previous two sections have depended on $\mathfrak{F}$ rather than f, the formation function defining locally, and where f has been mentioned it has not mattered whether

f has been taken to be integrated or not. However, for some of the results in this section this consideration is important, and so we shall assume for the rest of this chapter that (unless otherwise stated) f is integrated.

5.3.1 THEOREM. An $\mathfrak{F}$ - covering subgroup E of L relative to G covers those chief factors of G f-central for L. In particular, if M is a maximal subgroup of G containing E then M is f-abnormal for L.

Proof. We use induction on $|L|$. Let H/K be a chief factor of G f-central for L, write $C = C_L(H/K)$ and let M be a maximal subgroup of G $\mathfrak{F}$-crucial for L which contains the $\mathfrak{F}$-covering subgroup E of L relative to G. We have $L/C \in f(p) \leq \mathfrak{F}$, and therefore $R \leq C$ if R is the $\mathfrak{F}$-residual of L. Thus $MC \geq MR \geq ER = G$ by (5.2.3); hence $(L \cap M)C = L$, and $L \cap M/C_M(H/K) = L \cap M/C \cap M \cong L/C \in f(p)$. Since M is f-abnormal for L and H/K is f-central for L, by (2.3.6) M covers H/K, and therefore $H \cap M/K \cap M$ is a factor of M f-central for $L \cap M$. Since $|L \cap M| < |L|$ and E is an $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M, by induction E covers $H \cap M/K \cap M$ since this factor refines into chief factors of M each of which is f-central for $L \cap M$. Thus $H \cap M \leq E(K \cap M)$. If $H \leq M$, this means $H \leq EK$. If $H \nleq M$, then $G = HM$; but M covers H/K and therefore $G = KM$. Thus $H = K(H \cap M) \leq KE(K \cap M) = EK$. In either case E covers H/K as required. The last statement of (5.3.1) follows from (2.3.6).

<u>5.3.2</u> <u>DEFINITION</u>. In (2.3.5 (b)) we defined the concept of a maximal subgroup of G f-abnormal for L. We now extend this definition to an arbitrary subgroup H supplementing L in G. We say such a subgroup H is <u>f-abnormal for L</u> if, whenever $H \leq X^* \lessdot X \leq G$, then X^* is a maximal subgroup of X f-abnormal for $L \cap X$. This definition is clearly consistent with our earlier one. D.R. Taunt has shown in unpublished work that a subgroup Y of a soluble group Z is abnormal in the sense of Carter [4] if and only if every link of every maximal chain from Y up to Z is non-normal. Thus in the case $L = G$ and $f(p) = 1$ for all primes p our concept reduces to the usual concept of abnormality.

<u>5.3.3</u> <u>THEOREM</u>. G has subgroups E supplementing L which are f-abnormal for L and which satisfy $L \cap E \in \mathfrak{F}$, and these are precisely the $\mathfrak{F}$- covering subgroups of L relative to G.

<u>Proof</u>. Let E be an $\mathfrak{F}$- covering subgroup of L relative to G. By (5.2.5) E supplements L in G and satisfies $L \cap E \in \mathfrak{F}$. Suppose $E \leq X^* \lessdot X \leq G$. By (5.2.4) E is an $\mathfrak{F}$- covering subgroup of X relative to $L \cap X$, and therefore by (5.3.1) X^* is a maximal subgroup of X f-abnormal for $L \cap X$. Thus G has subgroups of the required form among which are the relative $\mathfrak{F}$- covering subgroups of L. We now show there are no others. Suppose H is a supplement of L in G which is f-abnormal for L and satisfies $L \cap H \in \mathfrak{F}$. If $HR \neq G$, then a maximal subgroup M of G containing HR complements some chief factor of G between L and R; but such chief factors are f-central for

L so that by (2.3.6) M is f-normal for L contradicting the f-abnormality of H for L. Hence $HR = G$. If $R = 1$, then $H = G$, and it is true that H is the unique relative $\mathfrak{F}$- covering subgroup of L in this case. Suppose then that $R \neq 1$, and let R/T be a chief factor of G, necessarily $\mathfrak{F}$-crucial for L. If $HT = G$, then $(L \cap H)T = L$, and therefore $L/T \cong L \cap H/T \cap H \in Q\mathfrak{F} = \mathfrak{F}$ by hypothesis. But this contradicts the definition of R, and therefore $HT \neq G$. Thus HT complements R/T in G and is therefore a maximal subgroup of G $\mathfrak{F}$-crucial for L. Now H is f-abnormal for $L \cap HT$ in HT, $H(L \cap HT) = HL \cap HT = HT$, and $L \cap HT \cap H = L \cap H \in \mathfrak{F}$. Since $|L \cap HT| < |L|$, by induction H is therefore an $\mathfrak{F}$-covering subgroup of $L \cap HT$ relative to HT. Because HT is $\mathfrak{F}$-crucial for L in G, H is therefore an $\mathfrak{F}$ - covering subgroup of L relative to G, as claimed.

We now prove a covering and avoidance theorem which is the analogue of a theorem for Carter subgroups proved by Carter in section 6 of [4]. We use the terminology of that paper.

5.3.4 THEOREM. Let E be an $\mathfrak{F}$ - covering subgroup of L relative to G. Then E covers the irreducible E-factors which are f-central for $L \cap E$ and avoids the rest. (If H/K is an E-factor then it is an elementary Abelian p-group for some prime p, and we say it is f-central for some $X \leq E$ if X induces on H/K an f(p) - group of automorphisms.)

<u>Proof</u>. Let H/K be an irreducible E-factor. Since $E \leq N_G(H)$ we may write $X = EH$. We use induction on $|L|$, distinguishing two cases:

<u>Case 1</u>. $X \neq G$. Since by (5.2.3) $LX = LEH = G$, we have $|L \cap X| < |L|$. By (5.2.4) E is an $\mathfrak{F}$ - covering subgroup of $L \cap X$ relative to X, and since H/K is an irreducible E-factor of X and $L \cap X \cap E = L \cap E$, the result follows at once by induction.

<u>Case 2</u>. $X = G$. In this case H/K is an irreducible EH - factor of G, that is a chief factor of G. Therefore either $EK = G$ or EK is a maximal subgroup of G complementing H/K. If E covers H/K and $EK = G$, then H/K is f-central for $L \cap E$; for $(L \cap E)K \triangleleft EK = G$, and so either $H \nleq (L \cap E)K$ in which case $L \cap E$ centralizes H/K, or $H \leq (L \cap E)K$ in which case $(L \cap E)K/K \cong L \cap E/L \cap E \cap K \in Q\mathfrak{F} = \mathfrak{F}$; but then by Clifford's Theorem (p.343 of [6]) H/K is the direct product of minimal normal f-central p-subgroups of $(L \cap E)K/K$, and is therefore f-central for $L \cap E$. Suppose, on the other hand, that E avoids H/K so that EK, $= M$ say, is a complement of H/K in G. By (5.3.1) M is f-abnormal for L, and therefore by (2.3.6) H/K is f-eccentric for L. But $LH/H \in \mathfrak{F}$; for if not, G has a chief factor $\mathfrak{F}$ -crucial for L between H and LH, and this is impossible since relative $\mathfrak{F}$ - covering subgroups avoid $\mathfrak{F}$-crucial chief factors whereas E covers LH/H. Thus H/K is $\mathfrak{F}$-crucial for L and we have

$$L \cap H / L \cap K \underset{G}{\cong} H / K . \qquad (*)$$

Thus $L \cap H/L \cap K$ is a chief factor of G f-eccentric for L. Since $L/L \cap H \in \mathfrak{F}$, we have $E(L \cap H) = G$ by (5.2.3), and therefore $(L \cap E)(L \cap H) = L \cap E(L \cap H) = L$. Thus $L/L \cap H$ is isomorphic with $L \cap E/L \cap E \cap H$ and since $L \cap H \leq C_L(L \cap H/L \cap K)$ the groups L and $L \cap E$ induce isomorphic groups of automorphisms on $L \cap H/L \cap K$. Since $L \cap H/L \cap K$ is f-eccentric for L, it is f-eccentric for $L \cap E$. Hence by (*) H/K is also f-eccentric for $L \cap E$. This completes the proof of the theorem.

We now consider briefly how the preceding results specialize when we take $L = G$. The relative $\mathfrak{F}$ - covering subgroups of L become the absolute $\mathfrak{F}$ - covering subgroups of the soluble group G. An $\mathfrak{F}$ -crucial maximal subgroup of G is one which complements an f-eccentric chief factor 'as near to G as possible' in some chief series of G, and the $\mathfrak{F}$ - covering subgroups are simply the minimal members of the $\mathfrak{F}$ -crucial maximal chains of G. Bearing in mind that $\mathfrak{F}$ - normalizers may be characterized similarly in terms of f-critical maximal chains and that an f-critical maximal subgroup is one which complements a chief factor 'as near to 1 as possible' in some chief series of G, one observes a certain duality between $\mathfrak{F}$ - covering subgroups and $\mathfrak{F}$ - normalizers. By (5.2.4) an $\mathfrak{F}$ - covering subgroup E of G is an $\mathfrak{F}$ - covering subgroup of every subgroup containing E, and by (5.2.5) E is characterized to within conjugacy by the two properties

(i) E belongs to $\mathfrak{F}$, and

(ii) E supplements the $\mathfrak{F}$ -residual of every subgroup containing E.

Perhaps in deference to the first of these properties Gaschütz called these subgroups the '$\mathfrak{F}$-subgroups' of G in [8]. Since we use the term $\mathfrak{F}$-subgroup in the wider sense to mean simply a subgroup belonging to the class $\mathfrak{F}$, we have adopted the alternative terminology '$\mathfrak{F}$- covering subgroup' to highlight the property (ii) above. Theorem 5.3.3 shows that the $\mathfrak{F}$- covering subgroups E are precisely the f-abnormal $\mathfrak{F}$- subgroups of G, and by (5.3.4) E covers those irreducible E-factors on which E induces an f-central group of automorphisms, and avoids the rest. With (4.4.9) in mind the duality mentioned above might lead us to consider the following conjectures:

1. The $\mathfrak{F}$ - covering subgroups of G are the minimal f-abnormal subgroups of G.
2. Let $\mathcal{E}$ be the set of $\mathfrak{F}$-subgroups of G which can be joined to G by an f-abnormal maximal chain; then the maximal members of $\mathcal{E}$ are precisely the $\mathfrak{F}$ - covering subgroups of G.

The first conjecture is easily disposed of. It is not difficult to verify that with $f(p) = 1$ for all primes p the subgroup H of Σ_4 which fixes the symbol 4 (and which is therefore isomorphic with Σ_3) is a minimal abnormal subgroup of Σ_4. But H is not nilpotent and is therefore not a Carter subgroup of Σ_4. Although it follows from (5.3.3) that the $\mathfrak{F}$ - covering subgroups of a group G are maximal members of the appropriate set $\mathcal{E}$, the converse is not true so that the second conjecture is also false. To show this we resort once again to the example discussed by Carter on page 562 of [5], (see (4.5.3)).

5.3.5 EXAMPLE. Let $G \cong C_5 \wr \Sigma_4$ with $G = NG^*$, $N = \langle a_1, a_2, a_3, a_4 \rangle$ the base group of order 5^4, $G^* \cong \Sigma_4$ and $N \cap G^* = 1$. Take $f(p) = 1$ for all primes p so that $\mathfrak{F} = \mathfrak{N}$. G has a Carter subgroup $E = N^*T$ where T is a Sylow 2-subgroup of G^* and $N^* = \langle a_1 a_2 a_3 a_4 \rangle$. Let H be the subgroup of G^* corresponding to the subgroup of Σ_4 which fixes the symbol 4. $M = NH$ is an abnormal maximal subgroup of G, and if g is the element of H corresponding to the transposition (12) of Σ_4, then M has a Carter subgroup $E^* = \langle g, a_1 a_2, a_3, a_4 \rangle$ of order 2.5^3 (and not 2.5^2 as stated by Alperin in [1]). E^* can be joined to M, and therefore to G, by a non-normal maximal chain, so E^* belongs to the set $\mathfrak{E}$ for G. We show E^* is a maximal member of $\mathfrak{E}$. For suppose $E^* \leq F \in \mathfrak{E}$; then F is nil-potent and $2.5^3 \mid |F|$. Suppose $5^4 \mid |F|$; then $N \leq F$ and $\langle g \rangle \leq C_G(N)$ which is impossible. Hence $\bar{N} = \langle a_1 a_2, a_3, a_4 \rangle$ is a Sylow 5-subgroup of F. Let S be a Sylow 5-complement of F so that S centralizes $\bar{N}$. Now $C_{G^*}(\bar{N}) = \langle g \rangle$, and therefore since $|\langle g \rangle| = 2$, we have $|F| = 2.5^3$ and $E^* = F$. Hence E^* is a maximal member of $\mathfrak{E}$ but is not an $\mathfrak{F}$ - covering subgroup of G.

We conclude this section with a result connecting the relative and absolute $\mathfrak{F}$ - covering subgroups of a group.

5.3.6 THEOREM. If E is an $\mathfrak{F}$ -covering subgroup of L relative to G then $L \cap E$ is an absolute $\mathfrak{F}$ - covering subgroup of L.

Proof. If $L \in \mathfrak{F}$ the theorem is true; for then $E = G$ and contains $L \cap E = L$ which by (5.1.5) is an absolute $\mathfrak{F}$ - covering subgroup of L. We therefore assume $L \notin \mathfrak{F}$ and use induction on $|L|$. Let R be the $\mathfrak{F}$-residual of L and M a maximal subgroup of G which contains E and which is $\mathfrak{F}$-crucial for L. Write $T = R \cap M$ so that by (5.1.4) R/T is a chief factor of G. Since $L \triangleleft G$ it follows from Clifford's Theorem that $\bar{R} = R/T$ decomposes into the direct product $\bar{R} = \bar{N}_1 \times \dots \times \bar{N}_r$ of minimal normal subgroups $\bar{N}_i = N_i/T$ of $\bar{L} = L/T$. It is clear from the definition of the $\mathfrak{F}$-residual of L that each $\bar{N}_i$ is an f-eccentric chief factor of L, and therefore that

$$(L \cap M) < (L \cap M)N_1 < (L \cap M)N_1N_2 < \dots < (L \cap M)N_1 \dots N_r = L$$

is an $\mathfrak{F}$-crucial maximal chain of L. Now E is an $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M, and since $|L \cap M| < |L|$ it follows by induction that $L \cap M \cap E = L \cap E$ is an absolute $\mathfrak{F}$ - covering subgroup of $L \cap M$. Hence $L \cap E$ can be joined to $L \cap M$, and therefore to L, by an $\mathfrak{F}$-crucial maximal chain. Thus $L \cap E$ is an $\mathfrak{F}$ - covering subgroup of L as required.

<u>5.4</u> In chapter three we developed a theory of $f(p)$ - normalizers which was distinct from, although in many ways analogous to, the theory of f - normalizers developed in the following chapter. We now discuss what might be naturally considered as a p-theory of $\mathfrak{F}$ - covering subgroups and show that in fact it turns out to be a special case of the general situation already dealt with in sections 5.1 - 5.3. We recall the $\mathfrak{F}_p$ denotes the p-local formation defined by $f(p)$.

We showed in section 5.1 that a maximal subgroup of G $\mathfrak{F}$-crucial for L is essentially a complement of a chief factor H/K of G below L which is f-eccentric for L and such that every chief factor of G between H and L is f-central for L. To obtain the corresponding p-theory it would seem natural to define a maximal subgroup of G $f(p)$-crucial for L as a complement of a p-chief factor H/K of G below L which is f-eccentric for L and such that every p-chief factor of G between H and L is f-central for L, or equivalently which is such that $L/K \notin \mathfrak{F}_p$ and $L/H \in \mathfrak{F}_p$. Now $\mathfrak{F}_p$ is a saturated formation defined locally by the formation function f^* where $f^*(p) = f(p)$ and $f^*(q) = \mathfrak{S}$ for $q \neq p$. Thus what we defined as a relative $f(p)$-crucial maximal subgroup turns out to be precisely a relative $\mathfrak{F}_p$-crucial maximal subgroup. To continue with the development of the p-theory we should then proceed to define an $f(p)$ - covering subgroup as a minimal member of a relative $f(p)$-crucial maximal chain, hoping to prove there was a unique such subgroup containing a given Sylow p-complement L^p of L. But, as we have just seen, this definition would yield none other than an $\mathfrak{F}_p$ - covering subgroup of L relative to G corresponding to any Sylow system of L containing L^p. The difference between this situation and that of $f(p)$ - normalizers is that whereas a relative $f(p)$-crucial maximal subgroup as also $\mathfrak{F}_p$-crucial, irrespective of whether or not f^* is an integrated formation function, it is not true that a relative $f(p)$-critical maximal subgroup is necessarily also a relative f^*-critical maximal subgroup for integrated f^* defining $\mathfrak{F}_p$.

We now pursue a little further our analogy with f - normalizers.

5.4.1 DEFINITION. Let f be a formation function defining the local formation $\mathfrak{F}$, and for each prime p let $\mathfrak{F}_p$ be the p-local formation defined by $f(p)$. If H_p is the $\mathfrak{F}_p$ - covering subgroup of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L, we define the local $\mathfrak{F}$ - covering subgroup H of L relative to G corresponding to $\mathfrak{S}$ by

$$H = \bigcap_{p \mid |L|} H_p .$$

The set of local $\mathfrak{F}$ - covering subgroups H forms a conjugacy class of G which is characteristic if $L \trianglelefteq G$, and if $f = Sf$ then $H \in \bigcap_p \mathfrak{F}_p = \mathfrak{F}$. Moreover, since $|G:H_p|$ is a power of p, the subgroups H_p (for different primes p) are pairwise permutable, and therefore if R/T is an irreducible H_p-factor of order a power of p then H covers R/T when R/T is f-central for $L \cap H_p$ and avoids it otherwise. It then follows from the homomorphism - invariance of the H_p that the class of local $\mathfrak{F}$- covering subgroups of L relative to G is also homomorphism - invariant. This line of investigation seems to be less rewarding than the corresponding investigation for f - normalizers; for as the following example shows the local $\mathfrak{F}$ - covering subgroups do not in general coincide with the standard $\mathfrak{F}$ - covering subgroups, even when f is an integrated, S-closed formation function. Let $G = C_5 \wr \Sigma_4$ as in (5.3.5), and take $f(5)$ = the class of 2-groups and $f(p) = 1$ otherwise. The formation function f is evidently S-closed and integrated, but in the notation of (5.3.5) NT is an $\mathfrak{F}$ - covering subgroup of G whereas $\langle g \rangle$ is a local $\mathfrak{F}$ - covering subgroup.

It would be of interest to know whether either the condition 'f = Sf' or the condition 'f is integrated' is sufficient to ensure that a local $\mathfrak{F}$ - covering subgroup is contained in a standard $\mathfrak{F}$ - covering subgroup. To see that this is not true in general, let G again be the group $C_5 \wr \Sigma_4$ and take $f(2) = \{Q, R_0, Sn\}\Sigma_3$, $f(5) = \{Q, R_0, Sn\}\Sigma_4$ and $f(p) = 1$ otherwise. Then in the notation of (5.3.5) NT is a local $\mathfrak{F}$ - covering subgroup of G whereas T is a standard $\mathfrak{F}$ - covering subgroup. This example also shows that local $\mathfrak{F}$ - covering subgroups need not belong to the formation $\mathfrak{F}$. It would also be of interest to know whether the conclusion that local $\mathfrak{F}$ - covering subgroups are $\mathfrak{F}$-groups is a consequence of the hypothesis that f is integrated.

5.5 We conclude this chapter with the second characterization of relative $\mathfrak{F}$ - covering subgroups mentioned at the outset.

5.5.1 LEMMA. If $L \in \mathfrak{N}\mathfrak{F}$ then the $\mathfrak{F}$ - normalizers and the $\mathfrak{F}$ - covering subgroups of L relative to G coincide.

Proof. If R is the $\mathfrak{F}$-residual of L , then by hypothesis $R \leq F(L)$, and therefore maximal subgroups of G $\mathfrak{F}$-crucial for L are also $\mathfrak{F}$-critical for L . Since the hypotheses carry over to such maximal subgroups, the result follows at once from (4.4.5) and the definition of relative $\mathfrak{F}$ - covering subgroups.

5.5.2 LEMMA. If $L \in \mathfrak{F}$ and H is a subgroup of G supplementing F(L) , then $L \cap H \in \mathfrak{F}$.

Proof. By induction on $|L|$. If $H = G$, the result is trivially true. If $H \neq G$, let M be a maximal subgroup of G containing H . Then M supplements $F(L)$, and by (4.3.9) $L \cap M \in \mathfrak{F}$. Now $F(L \cap M) \geq F(L) \cap M$, and therefore $H\,F(L \cap M) \geq H(F(L) \cap M) = H\,F(L) \cap M = M$. Therefore, since $|L \cap M| < |L|$, by induction we have $L \cap H = (L \cap M) \cap H \in \mathfrak{F}$ as required.

Our next lemma proves rather more than we shall need for the proof of the main theorem of this section, but the additional results are required in chapter six.

5.5.3 LEMMA. If $L \in \mathfrak{N}\mathfrak{F}$ and H is a subgroup of G satisfying $H\,F(L) = G$ and $L \cap H \in \mathfrak{F}$, then $L \cap N_G(H) \in \mathfrak{F}$ and $N_G(H)$ is contained in a unique $\mathfrak{F}$ - covering subgroup of L relative to G .

Proof. We use induction on $|L|$ to prove everything except uniqueness. Let N be a minimal normal subgroup of G contained in L . We have $F(L/N) \geq F(L)/N$, and therefore $HN/N \,.\, F(L/N) \geq H\,F(L)/N = G/N$. Moreover $(L/N) \cap (HN/N) = L \cap HN/N = (L \cap H)N/N \cong L \cap H/N \cap H \in Q\mathfrak{F} = \mathfrak{F}$, and $L/N \in \mathfrak{N}\mathfrak{F}$. Hence HN/N satisfies the hypotheses of the theorem for G/N , and therefore, since $|L/N| < |L|$, by induction $N_{G/N}(HN/N)$ is contained in an $\mathfrak{F}$- covering subgroup E^*/N of L/N relative to G/N . By (5.2.7) $E^* = EN$ for some $\mathfrak{F}$ - covering subgroup E of L relative to G . We distinguish two cases:

Case 1. $E^* \neq G$. By (5.2.4) E is an $\mathfrak{F}$ - covering subgroup of $L \cap E^*$ relative to E^* . Now $H \leq N_G(H) \leq N_G(HN) \leq E^*$, and since $F(L \cap E^*) \geq F(L) \cap E^*$ we have $E^* \geq H\,F(L \cap E^*) \geq H(F(L) \cap E^*) =$

$H\,F(L) \cap E^* = E^*$; we also have $L \cap E^*/F(L \cap E^*) \in Q(L \cap E^*/F(L) \cap E^*)$ $= Q(\ (L \cap E^*)F(L)/F(L)\ ) = Q(L/F(L)) \leq Q\mathfrak{F} = \mathfrak{F}$, and therefore $L \cap E^* \in \mathfrak{N}\mathfrak{F}$. Further, $H \cap L \cap E^* = H \cap L \in \mathfrak{F}$. Since E^* supplements L in G , we have $|L \cap E^*| < |L|$ and therefore by induction $\mathfrak{F} \ni L \cap E^* \cap N_{E^*}(H) = L \cap N_{E^*}(H)$ and $N_{E^*}(H)$ is contained in an $\mathfrak{F}$ - covering subgroup $\bar{E}$ of $L \cap E^*$ relative to E^* . Now $N_G(H) \leq E^*$, therefore $N_{E^*}(H) = N_G(H)$, and since by (5.2.2) $\bar{E}$ is a conjugate of E , we conclude that $\bar{E}$ is an $\mathfrak{F}$ - covering subgroup of L relative to G containing the $\mathfrak{F}$-group $L \cap N_G(H)$.

<u>Case 2</u>. $E^* = G$. If N is f-central for L then E covers N by (5.5.1) and (4.2.2). Thus $G = EN = E$ is an $\mathfrak{F}$ - covering subgroup of L relative to G ; $L \in \mathfrak{F}$, therefore by (5.5.2) $L \cap N_G(H)$ is in $\mathfrak{F}$ and the result is true. We may therefore assume $L \notin \mathfrak{F}$, that is $E \neq G$. It is well-known that a normal subgroup of a nilpotent group intersects the centre non-trivially; hence $N \leq Z(F(L))$ by the minimality of N . Then $E \cap F(L)$ is normalized by E , centralized by N and is therefore normal in $EN = G$. If $E \cap F(L) \neq 1$ we may choose a minimal normal subgroup N^* of G contained in $E \cap F(L)$, and apply the argument of case 1 with N^* replacing N ; for then $E^* = EN^* = E \neq G$. We are therefore left with the case $E \cap F(L) = 1$. But then $F(L) = EN \cap F(L) = (E \cap F(L))N = N$, and therefore $HN = G$. Since $L \notin \mathfrak{F}$, $H \neq G$ and H is a complement of N in G . We have $L/N = (L \cap E)N/N \cong L \cap E \in \mathfrak{F}$ and therefore by (5.1.7) all complements of N in G are conjugate. Hence H is a conjugate of E and is therefore an $\mathfrak{F}$ - covering subgroup

of L relative to G. Since N is f-eccentric for L, H is self-normalizing in G, and finally by (5.2.5 (b)) $L \cap H \in \mathfrak{F}$. Hence the theorem is true in this case and the induction argument is complete. To finish the proof of the lemma we need to show H is contained in a unique $\mathfrak{F}$ - covering subgroup of L relative to G, and this follows from

<u>5.5.4</u> <u>LEMMA</u>. If $L \in \mathfrak{N}\mathfrak{F}$, and E_1 and E_2 are $\mathfrak{F}$ - covering subgroups of L relative to G such that $(E_1 \cap E_2)\, F(L) = G$, then $E_1 = E_2$.

<u>Proof</u>. We use induction on $|L|$. Let N be a minimal normal subgroup of G contained in L. By (5.2.6) E_1N/N and E_2N/N are $\mathfrak{F}$ - covering subgroups of L/N ($\in \mathfrak{N}\mathfrak{F}$) relative to G/N. Also we have $E_1N/N \cap E_2N/N \geq (E_1 \cap E_2)N/N$ and $F(L/N) \geq F(L)/N$, and therefore $(E_1N/N \cap E_2N/N)\, F(L/N) = G/N$. Since $|L/N| < |L|$, by induction we have $E_1N = E_2N$. If N is f-central for L, then by (5.5.1) and (4.2.2) $N \leq E_1 \cap E_2$ and therefore $E_1 = E_2$. Thus we may assume that N and every minimal normal subgroup of G contained in L is f-eccentric for L and therefore avoided by E_i, $i = 1, 2$. If $G \neq E_1N = E_2N$, $= E^*$ say, then E_1 and E_2 are $\mathfrak{F}$ - covering subgroups of $L \cap E^*$ relative to E^* by (5.2.4). Now as in the proof of (5.5.3) we have $L \cap E^* \in \mathfrak{N}\mathfrak{F}$; we also have $E^* \geq (E_1 \cap E_2)\, F(L \cap E^*) \geq (E_1 \cap E_2)(F(L) \cap E^*) \geq (E_1 \cap E_2)\, F(L) \cap E^* = E^*$. Since $|L \cap E^*| < |L|$, we conclude by induction that $E_1 = E_2$. Finally, if $E^* = G$, then as in case 2 of the proof of (5.5.3) either G has a minimal normal subgroup contained in $E_1 \cap F(L)$,

or $E_1 \cap F(L) = 1$. Since the first possibility has already been excluded, the second holds, and $N = F(L)$. But then E_1 and E_2 are complements of N in G so that $(E_1 \cap E_2)N = G$ implies $|E_1 \cap E_2| = |E_1| = |E_2|$ and therefore $E_1 = E_2$ as required.

To prove the following main result we need from (5.5.3) only the weaker conclusion that H itself is contained in an $\mathfrak{F}$ - covering subgroup of L relative to G .

5.5.5 THEOREM. E is an $\mathfrak{F}$ - covering subgroup of L relative to G if and only if E is a supplement of L in G such that $\theta(E)$ is a subgroup of $\theta(G)$ maximal subject to the condition $\theta(E) \cap \theta(L) \in \mathfrak{F}$ for every homomorphism θ of G .

Proof. If E is an $\mathfrak{F}$ - covering subgroup of L relative to G, then by (5.2.5 (a)) E is a supplement of L in G , by (5.2.5 (b)) $L \cap E \in \mathfrak{F}$, and by (5.2.5 (c)) E is maximal subject to this condition. Hence by (5.2.6) the condition is necessary. To prove the sufficiency let E^* be a supplement of L in G such that $\theta(E^*)$ is maximal subject to the condition $\theta(E^*) \cap \theta(L) \in \mathfrak{F}$ for every homomorphism θ of G . We use induction on $|L|$. Let N be a minimal normal subgroup of G contained in L . The hypotheses clearly carry over to G/N , and since $|L/N| < |L|$ by induction E^*N/N is an $\mathfrak{F}$ - covering subgroup of L/N relative to G/N . Therefore by (5.2.7) we have $E^*N = EN$, $= M$ say, for a suitable $\mathfrak{F}$ - covering subgroup E of L relative to G . Since $L \cap M = L \cap EN = (L \cap E)N$ and $N \leq F(L \cap M)$, we have $(L \cap M)/F(L \cap M) \in Q(L \cap M/N) = Q(L \cap E/N \cap E) \leq$

$Q(L \cap E) \leq Q\mathfrak{F} = \mathfrak{F}$, since $L \cap E \in \mathfrak{F}$ by (5.2.5); hence $L \cap M \in \mathfrak{N}\mathfrak{F}$. Further $E^* F(L \cap M) \geq E^*N = M$, and $(L \cap M) \cap E^* = L \cap E^* \in \mathfrak{F}$ by hypothesis. Hence we can apply Lemma 5.5.3 with M , E^*, and $L \cap M$ playing the rôle of G , H , and L , and conclude that E^* is contained in an $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M . Since by (5.2.4) E is an $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M , by (5.2.2) E^* is therefore contained in a conjugate $\bar{E}$ of E . Hence, again by (5.2.2), $\bar{E}$ is an $\mathfrak{F}$ - covering subgroup of L relative to G , and therefore, since by (5.2.5) $\bar{E} \cap L \in \mathfrak{F}$, by maximality we have $\bar{E} = E^*$. This concludes the proof.

In the case $L = G$, $\mathfrak{F}$ - covering subgroups are characterized as the maximal $\mathfrak{F}$ -subgroups in every homomorphic image. This generalizes an unpublished result of Dr. J.S. Rose who has shown this to be a characterization of the Carter subgroups of a soluble group when $\mathfrak{F} = \mathfrak{N}$.

Chapter Six

RELATIONS BETWEEN $\mathfrak{F}$ - NORMALIZERS AND $\mathfrak{F}$ - COVERING SUBGROUPS

6.1 Our results on the theme of the title of this chapter first concern the inclusion relations between the two canonical conjugacy classes and criteria for their coincidence. We then look at connections between these classes in the whole group and the corresponding classes in special types of subgroups. We also investigate the embedding of $\mathfrak{F}$ - normalizers in $\mathfrak{F}$ - covering subgroups, although apart from Theorem 6.2.5 we obtain very little general information about this apparently difficult problem. Most of our positive results in this direction restrict the soluble normal subgroup L to a rather special subclass of soluble groups, and frequently generalize known theorems about the relationship between system normalizers and Carter subgroups. The chapter ends with a theorem and a counterexample about the covering and avoidance property of $\mathfrak{F}$ - normalizers. Except where otherwise stated f will denote an integrated formation function defining $\mathfrak{F}$ locally, and for the first three theorems f(p) denotes the formation defining the p-local formation $\mathfrak{F}_p$.

6.1.1 THEOREM. An $\mathfrak{F}_p$ - covering subgroup of L relative to G contains an f(p) - normalizer of L relative to G , and every relative f(p) - normalizer is contained in a relative $\mathfrak{F}_p$ - covering subgroup.

Proof. We use induction on $|L|$. If $L \in \mathfrak{F}_p$, G is both a relative

$f(p)$ - normalizer and $\mathfrak{F}_p$ - covering subgroup, so the result is true in this case. Now suppose $L \notin \mathfrak{F}_p$; let E be an $\mathfrak{F}_p$ - covering subgroup of L relative to G and M a maximal subgroup of G $\mathfrak{F}_p$-crucial for L which contains E. Since E is an $\mathfrak{F}_p$ - covering subgroup of $L \cap M$ relative to M and $|L \cap M| < |L|$, by induction E contains an $f(p)$ - normalizer $\bar{N}$ of $L \cap M$ relative to M. By (3.1.6) and the conjugacy of relative $f(p)$ - normalizers $\bar{N}$ contains an $f(p)$ - normalizer N of L relative to G. A fortiori E contains N and the first assertion of the theorem is proved; the second follows at once from the conjugacy property.

<u>6.1.2</u> <u>THEOREM</u>. If $L \in \mathfrak{N}^p \mathfrak{F}_p$, then the $f(p)$ - normalizers and the $\mathfrak{F}_p$ - covering subgroups of L relative to G coincide. In particular, they coincide when L has p-length one.

<u>Proof</u>. It is clear from (5.1.4) that a maximal subgroup M of G $\mathfrak{F}_p$-crucial for L supplements $O_{p'p}(L)$ in G, and is therefore $f(p)$-critical for L. It is easily verified that $L \cap M \in \mathfrak{N}^p \mathfrak{F}_p$ and the result now follows from (3.4.2) and (5.2.1).

<u>6.1.3</u> <u>THEOREM</u>. If $f(p) \leq \mathfrak{R}_p$, that is $\mathfrak{F}_p = \mathfrak{N}^p$, then the $f(p)$ - normalizers and the $\mathfrak{F}_p$ - covering subgroups of L relative to G coincide.

<u>Proof</u>. Let N be an $f(p)$ - normalizer of L relative to G ; then by (6.1.1) N is contained in an $\mathfrak{F}_p$ - covering subgroup E of L relative to G. Since N and E both cover G/L, it is sufficient to show the inclusion $L \cap N \leq L \cap E$ is in fact equality.

Since the group of automorphisms of an elementary Abelian p-group is well-known to have no non-trivial normal p-subgroups, a p-chief factor of G f-central for L is centralized by L; hence $C_p(L) = L$ and $L \cap N = N_L(L^p)$ for some Sylow p-complement L^p of L. But is it well-known that the normalizer of a Sylow p-complement is abnormal, and therefore $L \cap N \ntriangleleft L$. However, by (5.2.5) $L \cap E \in \mathfrak{N}^p$, and if $L \cap N$ were a proper subgroup of $L \cap E$ we should have a contradiction; for $|L \cap E : L \cap N|$ is a power of p and by a well-known property of $\mathfrak{N}^p$-groups a maximal subgroup of $L \cap E$ containing $L \cap N$ would be normal in $L \cap E$ contradicting the abnormality of $L \cap N$. Hence $L \cap E = L \cap N$ as required.

The following example shows that in general it is possible for the f(p) - normalizers, $\mathfrak{F}_p$ - normalizers and $\mathfrak{F}_p$ - covering subgroups all to be distinct.

6.1.4 EXAMPLE. Since 4 is the smallest integer n such that $5 \mid |GL(n,3)|$ it follows from Maschke's Theorem ([6], (10.8) on p.41) that the elementary Abelian group A of order 3^4 has an automorphism α of order 5 which acts faithfully and irreducibly. Let $B_i = A_i\langle\alpha_i\rangle$ be an isomorphic copy of the splitting extension of A by $\langle\alpha\rangle$, $i = 1, 2, 3$, and write $B = B_1 \times B_2 \times B_3$. Let H be a copy of Σ_3, and define G to be the splitting extension $BH \cong A\langle\alpha\rangle \wr \Sigma_3$ where H permutes the direct components B_i of the base group according to the standard permutation representation of Σ_3. Then $|G| = 2.3^{13}.5^3$. Let N be a minimal normal subgroup of G contained in $W = A_1 \times A_2 \times A_3$. If $1 \neq n \in N$, then we may write $n = a_1 a_2 a_3$ with

$a_i \in A_i$ and suppose without loss of generality that $a_1 \neq 1$. By the normality of N and the irreducibility of α we have $1 \neq a_1^{-1} a_1^{\alpha_1} = n^{-1} n^{\alpha_1} \in N$ and therefore $A_1 \leq N$. But H permutes the A_i transitively and therefore $W \leq N$. Thus W is a minimal normal subgroup of G. G/W has one central minimal normal subgroup $V/W = \langle \alpha_1 \alpha_2 \alpha_3 \rangle W/W$ of order 5 and a second U/W of order 5^2 which is a direct complement of V/W in B/W and on which H acts faithfully. Define $f(3)$ to be the class of $2'$-groups and let $\mathfrak{F}_3$ be the 3-local formation defined by $f(3)$. Then f^*, specified by $f^*(3) = f(3)$ and $f^*(q) = \mathfrak{F}_3$ for $q \neq 3$, is an integrated formation function defining $\mathfrak{F}_3$ locally. Let K be the normal Sylow 3-subgroup of H. G has two 3-chief factors, namely KB/B and $N/1$, in a suitable chief series, and both are f-eccentric. Thus the $f(3)$ - normalizers of G have order 2.5^3 and are precisely the Sylow 3-complements of G. Of the 5-chief factors in a particular chief series, UV/V is f^*-eccentric and V/W is f^*-central, and therefore an $\mathfrak{F}_3$- normalizer of G has order 2.5. If X is the subgroup of H generated by the element corresponding to the transposition $(12) \in \Sigma_3$, then $M = XB$ is an $\mathfrak{F}_3$-crucial maximal subgroup of G. Now A_3 is a minimal normal subgroup of M with $\mathrm{Aut}_M(A_3) \cong C_5$, and therefore $A_3/1$ is an f-central chief factor of M. It is easily verified by an argument similar to the one used above that $A_1 \times A_2$ is a minimal normal subgroup of M with $|\mathrm{Aut}_M(A_1 \times A_2)| = 2.5$, and therefore $A_1 \times A_2/1$ is an f-eccentric 3-chief factor of M. Thus $A_3(\langle\alpha_1\rangle \times \langle\alpha_2\rangle \times \langle\alpha_3\rangle)X$ is an $\mathfrak{F}_3$- covering subgroup of M and therefore also of G since M

is $\mathfrak{F}_3$-crucial in G . Hence the three canonical conjugacy classes mentioned above are distinct in this group G .

This section ends with the analogue of (6.1.1) for the general situation.

6.1.5 THEOREM. If f is either an integrated or an S-closed formation function defining $\mathfrak{F}$ locally and $\mathfrak{S}$ a Sylow system of L ; and if D and E are respectively the f - normalizer and $\mathfrak{F}$ - covering subgroup of L relative to G corresponding to $\mathfrak{S}$, then $D \leq E$.

Proof. We use induction on $|L|$. If $L \in \mathfrak{F}$ the relative f - normalizers and $\mathfrak{F}$ - covering subgroups coincide with G and the result is true. Therefore suppose $L \notin \mathfrak{F}$ and let M be an $\mathfrak{F}$-crucial maximal subgroup of G such that $\mathfrak{S}$ reduces into $L \cap M$. By definition E is the $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$, and since $|L \cap M| < |L|$ by induction E contains the f - normalizer $\bar{D}$ of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$. By (4.4.1) if f is integrated and by (4.5.2) if $f = Sf$ $\bar{D}$ contains D and the proof is complete.

6.2 In this section we are concerned mainly with the conditions for coincidence of $\mathfrak{F}$ - covering subgroups with f - normalizers. Carter shows in [4] that a system normalizer is a Carter subgroup of G if and only if it is self-normalizing in G , or equivalently if and only if it is abnormal in G . However for a general $\mathfrak{F}$ - normalizer D the condition $D = N_G(D)$, or even the stronger condition $D \rtimes G$ is inadequate to ensure that D is an $\mathfrak{F}$ - covering subgroup of G, even when $\mathfrak{F} = \mathfrak{U}$, the class of supersoluble groups. This is demonstrated by

6.2.1 EXAMPLE. This is a simple modification of Example 5.3.5 whose notation we continue to use here. Let α be the automorphism of N specified by the mapping $a_i \longrightarrow a_i^2$ $(i = 1, \ldots, 4)$ of the generators of N. Then α has order 4 and commutes elementwise with G^* so that $G^* \times <\alpha> \leq \mathrm{Aut}(N)$. Let W be the splitting extension of N by $G^* \times <\alpha>$. By considering the f-critical and $\mathfrak{U}$-crucial chains of G (bearing in mind that $f(p) = \mathfrak{A}$-groups of exponent $p - 1$ for all primes p) it is easy to verify that $D = < a_1 a_2 a_3 a_4 > H < \alpha >$ is a $\mathfrak{U}$- normalizer of G contained in the $\mathfrak{U}$- covering subgroup $E = (< a_1 a_2 a_3 > \times < a_4 >) H < \alpha >$. We show that $D \rtimes G$. By the unpublished result of Taunt cited in (5.3.2) it will be sufficient to show that every link of every maximal chain from D to G is non-normal. We first observe that there are precisely two maximal subgroups of G containing D, namely $M_1 = < a_1 a_2 a_3 a_4 > G^* < \alpha >$ and $M_2 = NH < \alpha >$, and that both are non-normal in G. D is a $\mathfrak{U}$-covering subgroup of M_1, and is therefore abnormal in M_1; moreover D contains the Carter subgroup $< g, \alpha >$ of M_2, and is therefore abnormal in M_2. Hence D is an abnormal $\mathfrak{U}$- normalizer but not a $\mathfrak{U}$- covering subgroup of G.

Still in search of a criterion for $\mathfrak{F}$- normalizers to coincide with $\mathfrak{F}$- covering subgroups we formulate Carter's condition in a different way.

"The system normalizer D is a Carter subgroup of G if and only if whenever $D \lessdot X \leq G$ we have D non-normal in X."
This is clearly equivalent to the condition $D = N_G(D)$ but the reformulation extends naturally to the general situation.

<u>6.2.2</u> <u>THEOREM</u>. A necessary and sufficient condition for an $\mathfrak{F}$-normalizer of L relative to G to be an $\mathfrak{F}$-covering subgroup of L relative to G is that whenever $D \lessdot X \leq G$ then D is a maximal subgroup of X f-abnormal for $L \cap X$.

In the statement of (6.2.2) f is of course an integrated function. In (5.3.3) we have already proved the necessity of the condition, and in order to deal with the sufficiency we first prove

<u>6.2.3</u> <u>THEOREM</u>. Let E be the $\mathfrak{F}$-covering subgroup of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L, and let X be a subgroup which can be joined to G by a maximal chain of the form

$$X = X_r \lessdot X_{r-1} \lessdot \quad \dots \quad \lessdot X_0 = G . \qquad (+)$$

where $X_i\, F(L \cap X_{i-1}) = X_{i-1}$ and $\mathfrak{S} \cap X_{i-1}$ reduces into $L \cap X_i$, $i = 1, 2, \dots, r$; then $E \cap X$ is the $\mathfrak{F}$-covering subgroup of $L \cap X$ relative to X corresponding to $\mathfrak{S} \cap X$.

<u>Proof</u>. We use induction on $|L|$. It is clearly sufficient to prove that $E \cap X_1$ is the $\mathfrak{F}$-covering subgroup of $L \cap X_1$ relative to X_1 corresponding to $\mathfrak{S} \cap X_1$. Since $(L \cap X_1)\, F(L) = L \cap X_1 F(L) = L$, we have

$$L / F(L) \underset{X_1}{\cong} L \cap X_1 / F(L) \cap X_1 \qquad (*)$$

Hence if $L \in \mathfrak{N}\mathfrak{F}$, then $L \cap X_1 \in \mathfrak{N}\mathfrak{F}$, and by (5.5.1) the relative

$\mathfrak{F}$ - normalizers and $\mathfrak{F}$ - covering subgroups of L, and also of $L \cap X_1$, coincide. In this case the result follows from (4.4.2). We may therefore suppose that $L/F(L) \notin \mathfrak{F}$ so that by (*) X_1 has a chief factor S/T between $F(L) \cap X_1$ and $L \cap X_1$ which is $\mathfrak{F}$-crucial for $L \cap X_1$. Let M be the complement of S/T in X_1 which is such that $\mathfrak{G} \cap X$ reduces into $L \cap M$; then by (*) $M^* = M\,F(L)$ is a maximal subgroup of G $\mathfrak{F}$-crucial for L which is such that $\mathfrak{G}$ reduces into $L \cap M^*$ (compare Corollary 2.8 of [3]). Since $M = M^* \cap X_1$ supplements $F(L \cap M^*)$ $(\geq F(L))$ in M^*, and since $|M^* : M| = |G : X|$ is a power of a prime, M can be joined to M^* by a chain of the form (+) into which $\mathfrak{G} \cap M^*$ reduces (compare Lemma 2.5 of [3]). Since by definition E is the $\mathfrak{F}$ - covering subgroup of $L \cap M^*$ relative to M^* corresponding to $\mathfrak{G} \cap M^*$, and since $|L \cap M^*| < |L|$, by induction $\bar{E} = E \cap M$ is the $\mathfrak{F}$- covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{G} \cap M$. But since M is a maximal subgroup of X_1 $\mathfrak{F}$-crucial for $L \cap X_1$ and $\mathfrak{G} \cap X_1$ reduces into $L \cap M$, $\bar{E}$ is the $\mathfrak{F}$- covering subgroup of $L \cap X_1$ relative to X_1 corresponding to $\mathfrak{G} \cap X_1$. Since $E \cap M = E \cap M^* \cap X_1 = E \cap X_1$, the proof of the theorem is complete.

<u>6.2.4</u> <u>DEFINITION</u>. If X is a subgroup supplementing L in G and contained in the subgroup Y, we say X is <u>f-subnormal for L</u> in Y if there is a maximal chain $X = X_r \lessdot X_{r-1} \lessdot \dots \lessdot X_0 = Y$ such that X_i is a maximal subgroup of X_{i-1} f-normal for $L \cap X_{i-1}$, $i = 1, 2, \dots, r$. We use the notation <u>$X \triangleleft f \triangleleft Y$ for L</u> to describe

this relation.

The sufficiency of the criterion of (6.2.2) is now an easy consequence of the following proposition.

6.2.5 THEOREM. Let D and E be respectively the $\mathfrak{F}$- normalizer and $\mathfrak{F}$ - covering subgroup of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L; then $D \triangleleft f \triangleleft E$ for L.

Proof. We use induction on $|L|$, bearing in mind that by (6.1.5) $D \leq E$. If $D = E$ there is nothing to prove; therefore by (5.5.1) we may assume $L \notin \mathfrak{M}\mathfrak{F}$. Let M be a maximal subgroup of G f-critical for L which is such that $\mathfrak{S}$ reduces into $L \cap M$. By (4.4.4) and (6.2.3) D and $E^* = E \cap M$ are respectively the $\mathfrak{F}$ - normalizer and $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$. Since $|L \cap M| < |L|$, by induction we have $D \triangleleft f \triangleleft E^*$ for $L \cap M$ (and therefore for L); therefore if $E^* = E$ the theorem is true. To complete the proof we assume $E^* < E$ and show that $E^* \triangleleft f \triangleleft E$ for L. Let p be the prime dividing $|G : M|$; since M is f-critical for L, M supplements in G the Sylow p-subgroup P of $F(L)$. By (5.2.6) EP/P is an $\mathfrak{F}$ - covering subgroup of L/P relative to G/P, and by the isomorphism $M/P \cap M \cong G/P$ so also is E^*P/P. Hence $E^*P = EP$ and we have $1 \neq |E : E^*| = |E^*(P \cap E) : E^*| = |P \cap E : P \cap E^*|$. By (2.3.8) $P \cap M \triangleleft G$ and therefore $P \cap E^* = (P \cap M) \cap E \triangleleft E$. Let $P \cap E^* = P_r < P_{r-1} < \dots$ $\dots < P_0 = P \cap E$ be a chief series of E. Then for $i = 1, 2, \dots, r$ since P_{i-1}/P_i is a minimal normal p-subgroup of E/P_i contained in the normal p-subgroup $P \cap E/P_i$, by a well-known theorem $P \cap E \leq C_E(P_{i-1}/P_i)$; therefore E^*P_{i-1} and $E^*(P \cap E) = E$ induce isomorphic groups of automorphisms on P_{i-1}/P_i as do also $(L \cap E^*)P_{i-1}$ and $(L \cap E^*)(P \cap E) = L \cap E$. Thus if we write $E^*P_i = X_i$, then P_{i-1}/P_i is a p-chief factor of X_{i-1} supplemented in X_{i-1} by X_i , and we have a chain of subgroups $E^* = X_r < X_{r-1} < \dots < X_o = E^*P_o = E$. Since by (5.2.5 (b)) $L \cap E \in \mathfrak{F}$, P_{i-1}/P_i is f-central for $L \cap E$, and therefore for $(L \cap E^*)P_{i-1} = L \cap X_{i-1}$. Therefore by (2.3.6) X_i is a maximal subgroup of X_{i-1} f-normal for $L \cap X_{i-1}$, $i = 1, 2, \dots, r$, and hence $E^* \triangleleft f \triangleleft E$ as claimed. The proof of this theorem and therefore also of (6.2.2) is now complete.

While we have the notation of this proof at hand we make an observation pertinant to part of the proof of our next result. Let R be the $f(p)$ - residual of $L \cap D$. Since $L \cap D \in \mathfrak{F}$ by (4.4.6) , R is p-nilpotent and therefore has a unique Sylow p-complement, R^p say. Now suppose that $D = E^*$ in the above situation, so that by our previous remarks we have $\mathrm{Aut}_{L \cap D}(P_{i-1}/P_i) = \mathrm{Aut}_{L \cap E}(P_{i-1}/P_i) \in f(p)$, and therefore $R^p \leq C_E(P_{i-1}/P_i)$, $i = 1, 2, \dots, r$. In this case every p-chief factor of $R^p(P \cap E)$ is central; therefore $R^p(P \cap E)$ is p-nilpotent and $[R^p,(P \cap E)] = 1$. Then if $|E : D| = |E : E^*| \neq 1$, we have $P \cap E > P \cap D$, and therefore $C_{L_p}(R^p) \nleq D$ for any Sylow p-subgroup L_p of L . We use this fact in proving another criterion for the coincidence of relative $\mathfrak{F}$ - normalizers with relative $\mathfrak{F}$ - covering subgroups.

6.2.6 THEOREM. Let D be the $\mathfrak{F}$ - normalizer of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L ; let R^p denote the Sylow p-complement of the $f(p)$ - residual of $L \cap D$ and L_p the Sylow p-subgroup of L in $\mathfrak{S}$. Then the following conditions are equivalent:

(a) D is an $\mathfrak{F}$ - covering subgroup of L relative to G ;

(b) $N_{L_p}(R^p) \leq D$ for all primes p dividing $|L|$;

(c) $C_{L_p}(R^p) \leq D$ for all primes p dividing $|L|$.

Proof. (a) $\Rightarrow$ (b). We use induction on $|L|$. Suppose D is both an $\mathfrak{F}$ - normalizer and an $\mathfrak{F}$ - covering subgroup of L relative to G , and without loss of generality assume $D \neq G$. Let $p \mid |L|$ and write $N = N_G(R^p)$. Since $D \leq N$, by (5.2.4) D is an $\mathfrak{F}$ - covering subgroup of $L \cap N$ relative to N . By (5.3.6) $L \cap D$ is an absolute $\mathfrak{F}$ - covering subgroup of L and therefore by (5.3.3) $L \cap D \rtimes L$. By the well-known result that a Sylow system reducing into an abnormal subgroup also reduces into every subgroup which contains it, since $\mathfrak{S}$ reduces into $L \cap D$, $\mathfrak{S}$ also reduces into $L \cap N$. Therefore if $N \neq G$, we have $|L \cap N| < |L|$, and by induction it follows that $N_{L_p}(R^p) = N_{L_p \cap N}(R^p) \leq D$, because $L_p \cap N$ is the Sylow p-subgroup of $L \cap N$ in $\mathfrak{S} \cap N$. Hence we may suppose $R^p \triangleleft G$. Let M be a maximal subgroup of G $\mathfrak{F}$ -crucial for L which is such that $\mathfrak{S}$ reduces into $L \cap M$. Then D is the $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$. Suppose p is the prime dividing $|G : M|$; we show this gives a contradiction.

6.2.6 THEOREM. Let D be the $\mathfrak{F}$ - normalizer of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L ; let R^p denote the Sylow p-complement of the f(p) - residual of $L \cap D$ and L_p the Sylow p-subgroup of L in $\mathfrak{S}$. Then the following conditions are equivalent:

(a) D is an $\mathfrak{F}$ - covering subgroup of L relative to G ;

(b) $N_{L_p}(R^p) \leq D$ for all primes p dividing $|L|$;

(c) $C_{L_p}(R^p) \leq D$ for all primes p dividing $|L|$.

Proof. (a) $\Rightarrow$ (b). We use induction on $|L|$. Suppose D is both an $\mathfrak{F}$ - normalizer and an $\mathfrak{F}$ - covering subgroup of L relative to G , and without loss of generality assume $D \neq G$. Let $p \mid |L|$ and write $N = N_G(R^p)$. Since $D \leq N$, by (5.2.4) D is an $\mathfrak{F}$ - covering subgroup of $L \cap N$ relative to N . By (5.3.6) $L \cap D$ is an absolute $\mathfrak{F}$ - covering subgroup of L and therefore by (5.3.3) $L \cap D \rtimes L$. By the well-known result that a Sylow system reducing into an abnormal subgroup also reduces into every subgroup which contains it, since $\mathfrak{S}$ reduces into $L \cap D$, $\mathfrak{S}$ also reduces into $L \cap N$. Therefore if $N \neq G$, we have $|L \cap N| < |L|$, and by induction it follows that $N_{L_p}(R^p) = N_{L_p \cap N}(R^p) \leq D$, because $L_p \cap N$ is the Sylow p-subgroup of $L \cap N$ in $\mathfrak{S} \cap N$. Hence we may suppose $R^p \triangleleft G$. Let M be a maximal subgroup of G $\mathfrak{F}$ -crucial for L which is such that $\mathfrak{S}$ reduces into $L \cap M$. Then D is the $\mathfrak{F}$ - covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$. Suppose p is the prime dividing $|G : M|$; we show this gives a contradiction.

Under this assumption M complements a p-chief factor H/K of G below L such that $L/H \in \mathfrak{F}$ and $L/K \notin \mathfrak{F}$. But $(L \cap D)H = L \cap DH = L \cap G = L$ by (5.2.3), and therefore $Aut_{L \cap D}(H/K) \cong Aut_L(H/K)$. But since $R^p \triangleleft G$ and $p \nmid |R^p|$ we have $R^p \leq C_L(H/K)$. Now it is well-known that $Aut_L(H/K)$ has no non-trivial normal p-subgroups; therefore, since $L \cap D/R^p \in \mathfrak{R}_p f(p)$, we have $Aut_{L \cap D}(H/K) \in f(p)$ and H/K is f-central for $L \cap D$. Hence H/K is f-central for L and we have a contradiction; for by hypothesis H/K is $\mathfrak{F}$-crucial and therefore certainly f-eccentric for L. Hence $p \nmid |G : M|$, $L_p \leq M$, and therefore, since $|L \cap M| < |L|$, by induction we have $N_{L_p}(R^p) \leq N_{L_p \cap M}(R^p \cap M) \leq D$ as required.

Since it is clear that <u>(b) $\Rightarrow$ (c)</u>, it remains to show that <u>(c) $\Rightarrow$ (a)</u>. We assume that D is properly contained in the $\mathfrak{F}$-covering subgroup E of L relative to G, and use induction on $|L|$ to prove that $C_{L_p}(R^p) \nleq D$ where L_p belongs to the Sylow system $\mathfrak{S}$ of L to which E corresponds. Our assumption implies $D \neq G$, and so we can take a maximal subgroup M of G f-critical for L such that $\mathfrak{S}$ reduces into $L \cap M$. By (4.4.4) D is the $\mathfrak{F}$-normalizer of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$, and by (6.1.5) and (6.2.3) $D \leq E^* = M \cap E$ which is the $\mathfrak{F}$-covering subgroup of $L \cap M$ relative to M corresponding to $\mathfrak{S} \cap M$. If $D < E^*$, since $|L \cap M| < |L|$, by induction we have $C_{L_p \cap M}(R^p \cap M) \nleq D$ for some prime $p \mid |L \cap M|$. But $R^p \leq L \cap D \leq M$ and so we then have $C_{L_p \cap M}(R^p \cap M) \leq C_{L_p}(R^p) \nleq D$ as required. If on the other hand $D = E^*$, the result follows at once from the observations immediately preceding the statement of the theorem. This completes the proof.

6.3 The first result of this section is a simple corollary of (6.2.3).

6.3.1 THEOREM. Let f and f^* be integrated formation functions deining the local formations $\mathfrak{F}$ and $\mathfrak{F}^*$ respectively. Let D^* and E be respectively the $\mathfrak{F}^*$ - normalizer and the $\mathfrak{F}$ - covering subgroup of L relative to G corresponding to the Sylow system $\mathfrak{S}$ of L. Then $E \cap D^*$ is the $\mathfrak{F}$ - covering subgroup of $L \cap D^*$ relative to D^* corresponding to $\mathfrak{S} \cap D^*$.

Proof. By (4.4.5) D^* is the minimal member of a relative f-critical chain into which $\mathfrak{S}$ reduces, and this is precisely a chain of the form (+) in (6.2.3). Theorem 6.3.1 is therefore simply a special case of (6.2.3) with D^* in place of X.

This theorem has some interesting applications. For example suppose $\mathfrak{F}^* \leq \mathfrak{F}$; in this case $L \cap D^* \in \mathfrak{F}$, and therefore $D^* = E \cap D^*$, that is $D^* \leq E$. This is in contrast to the fact that in general it is not true that $E^* \leq E$ for some $\mathfrak{F}^*$ - covering subgroup E^* of L relative to G . Because if we take for example $L = G = \Sigma_4$, and $\mathfrak{F}^* = \mathfrak{N} \leq \mathfrak{F} = \mathfrak{U}$; then the $\mathfrak{N}$ - covering subgroups (Carter subgroups) of G have order 8 , whereas the $\mathfrak{U}$ - covering subgroups have order 6 . As a second application let $\mathfrak{F}^* = \mathfrak{N}\mathfrak{F}$. Since by (4.4.6) we have $L \cap D^* \in \mathfrak{F}^* = \mathfrak{N}\mathfrak{F}$, then by (5.5.1) the $\mathfrak{F}$ - covering subgroup $E \cap D^*$ is an $\mathfrak{F}$ - normalizer of $L \cap D^*$ relative to D^*, and therefore by an obvious extension of (4.6.1) to the 'relative' case $E \cap D^*$ is an $\mathfrak{F}$ - normalizer of L relative to G. It follows as a special case that the intersection of a Carter subgroup of a soluble group G with a suitable supersoluble normalizer of G is a system normalizer of G.

As a further application of (6.2.3) we prove

6.3.2 THEOREM. Let G be a soluble group with a normal Hall ϖ-subgroup K, and let $\mathfrak{S}$ be a Sylow system of G containing the Hall ϖ-complement H of G. If E is the $\mathfrak{F}$- covering subgroup of G corresponding to $\mathfrak{S}$, then $E \cap H$ is the $\mathfrak{F}$ - covering subgroup of H corresponding to $\mathfrak{S} \cap H$.

Proof. By (6.2.3) it is sufficient to prove the contention that H may be joined to G by a maximal chain of the form

$$H = X_r \lessdot X_{r-1} \lessdot \dots \lessdot X_0 = G ,$$

with $H_i F(H_{i-1}) = H_{i-1}$, $i = 1, 2, \dots ,r$, and such that $\mathfrak{S}$ reduces into each member of the chain. We prove this by induction on $|G|$. If $K = 1$, the result is trivial. Therefore assume $K \neq 1$. If $K \leq \Phi(G)$, we have $H\Phi(G) \geq HK = G$, and therefore $H = G$ which has been ruled out. Therefore $K > K \cap \Phi(G)$, $= K^*$ say, and we may choose a minimal normal subgroup N/K^* of G/K^* contained in K/K^*. By (2.4.3) $N \leq F(G)$, and since N/K^* is complemented in G we may choose a complement X_1 into which $\mathfrak{S}$ reduces. If $p \mid |G : X_1|$ we have $p \in \varpi$, and therefore $H \leq S^p \leq X_1$ where S^p is the Sylow p-complement of $\mathfrak{S}$ (compare Lemma 2.5 of [3]). Moreover X_1 has a normal Hall ϖ-subgroup $K \cap X_1$, and therefore as $|X_1| <$

$|G|$ our contention now follows by induction.

In exactly the same way we may apply (4.4.3) to prove

6.3.3 THEOREM. Let G, H and $\mathfrak{S}$ be as in (6.3.2) and let D be the $\mathfrak{F}$ - normalizer of G corresponding to $\mathfrak{S}$. Then $D \cap H$ is the $\mathfrak{F}$ - normalizer of H corresponding to $\mathfrak{S} \cap H$.

6.4 In this section we discuss some special results for the case when the soluble normal subgroup L of G belongs to the class $\mathfrak{N}\mathfrak{N}\mathfrak{F}$. First we give a simple proof of a result which in the 'absolute' case is due to R.W.Carter.

6.4.1 THEOREM. If $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$, each $\mathfrak{F}$ - normalizer D of L relative to G is contained in a unique $\mathfrak{F}$ - covering subgroup E of L relative to G.

Proof. By induction on $|L|$. Suppose D is contained in two relative $\mathfrak{F}$ - covering subgroups E_1 and E_2. Since $L/F(L) \in \mathfrak{N}\mathfrak{F}$, by (5.5.1) and by the homomorphism-invariance of the two canonical conjugacy classes (see (4.2.4) and (5.2.6)) we have $D\,F(L) = E_1F(L) = E_2F(L)$, $= X$ say. By (5.2.4) E_i is an $\mathfrak{F}$- covering subgroup of $L \cap X$ relative to X, and since $L \cap D \in \mathfrak{F}$ we have $L \cap X$ $(= L \cap D\,F(L) = (L \cap D)\,F(L)) \in \mathfrak{N}\mathfrak{F}$. Hence we can apply (5.5.3) with $H = D$ to deduce $E_1 = E_2$ as required.

6.4.2 DEFINITION. Let X be a subgroup supplementing L in G and Y a subgroup satisfying

(i) $X \triangleleft f \triangleleft Y$ for L, and

(ii) whenever $X \triangleleft f \triangleleft Z$ for L, then $Z \leq Y$.

Then we say Y is the <u>f-subnormalizer of X for L</u>. When $f(p) = 1$ for all primes p and $L = G$, it is clear that this concept is consistent with the definition of a subnormalizer given by Carter in section 3 of [5].

<u>6.4.3</u> <u>LEMMA</u>. Let $L \in \mathfrak{N}\mathfrak{F}$ and let H be a supplement of $F(L)$ in G such that $L \cap H \in \mathfrak{F}$. If X contains H as a maximal subgroup f-normal for $L \cap X$, then $L \cap X \in \mathfrak{F}$.

<u>Proof</u>. We use induction on $|L|$. First suppose $X \neq G$; we have $H\,F(L \cap X) \geq H(F(L) \cap X) = H\,F(L) \cap X = X$; moreover $(L \cap X) \cap H = L \cap H \in \mathfrak{F}$. Hence as $|L \cap X| < |L|$, by induction $L \cap X \in \mathfrak{F}$. Therefore we may assume $X = G$, that is, H is a maximal subgroup of G f-normal for L and supplementing $F(L)$. Since $L \cap H \in \mathfrak{F}$, H is an $\mathfrak{F}$-covering subgroup of $L \cap H$ relative to H; hence by (6.2.3) H is contained in an $\mathfrak{F}$-covering subgroup E of L relative to G. Since by (5.3.3) $E = H$ would imply H f-abnormal for L, we must have $E = G$. But then $L \cap X = L \cap G = L \in \mathfrak{F}$ as required.

Repeated application of this result gives

<u>6.4.4</u> <u>LEMMA</u>. If L and H satisfy the hypotheses of (6.4.3), and if $H \triangleleft f \triangleleft H^*$ for L, then $L \cap H^* \in \mathfrak{F}$.

<u>6.4.5</u> <u>THEOREM</u>. If $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$, and D is an $\mathfrak{F}$-normalizer of L relative to G contained in the $\mathfrak{F}$-covering subgroup E of G,

then E is the f-subnormalizer of D for L .

Proof. By (6.2.5) $D \triangleleft f \triangleleft E$ for L so it remains to show that condition (ii) of Definition 6.4.2 holds. To this end suppose $D \triangleleft f \triangleleft Z$ for L . It is clear that 'f-subnormality for L' is a homomorphism-invariant property, and therefore writing $K = F(L)$ we have $DK/K \triangleleft f \triangleleft ZK/K$ for L/K . Since $L/K \in \mathfrak{N}\mathfrak{F}$, we have $DK/K = EK/K$ is an $\mathfrak{F}$ - covering subgroup of L/K relative to G/K by (4.2.4), (5.2.6) and (5.5.1), and hence by (5.3.3) DK/K is f-abnormal for L/K in G/K . Thus $DK = ZK$, $= D^*$ say. Since $(L \cap D^*) \cap D = L \cap D \in \mathfrak{F}$, $K \leq F(L \cap D^*)$ and therefore $L \cap D^* \in \mathfrak{N}\mathfrak{F}$, we may apply (6.4.4) with $L \cap D^*$ and D^* in place of L and G , and conclude that $L \cap Z \in \mathfrak{F}$. Hence by (5.5.3) Z is contained in E^*, an $\mathfrak{F}$-covering subgroup of L relative to G . Since $D \leq Z \leq E^*$, by (6.4.1) $E^* = E$. Thus $Z \leq E$, condition (ii) holds and our proof is complete.

As another consequence of (5.5.3) we have

6.4.6 THEOREM. If $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$, and if D_1 and D_2 are $\mathfrak{F}$ - normalizers of L relative to G contained in the same $\mathfrak{F}$ - covering subgroup E of L relative to G , then D_1 and D_2 are conjugate in E .

Proof. Write $N = N_G(D_1)$ and $K = F(L)$. Then $NK/K \leq N_{G/K}(D_1K/K) = N_{G/K}(EK/K)$. But by (5.3.3) EK/K is self-normalizing in G/K , and therefore $NK \leq D_1K$, $= X$ say. Since $L \cap D_1 \in \mathfrak{F}$, clearly $L \cap X = L \cap D_1K = (L \cap D_1)K \in \mathfrak{N}\mathfrak{F}$; further we have $D_1F(L \cap X) = X$ and $(L \cap X) \cap D_1 = L \cap D_1 \in \mathfrak{F}$. Therefore Theorem 5.5.3 applies with D_1 in the rôle of H and we conclude that N is contained in some $\mathfrak{F}$ - covering subgroup E^* of L relative to G. By (6.4.1) $E^* = E$ and therefore $N_G(D_1) = E \cap N_G(D_1) = N_E(D_1)$. Since each relative $\mathfrak{F}$ - normalizer of L is contained in precisely one relative $\mathfrak{F}$ - covering subgroup of L, the number of $\mathfrak{F}$ - normalizers of L relative to G contained in E is $|G : N_G(D_1)|/|G : E|$ $= |G : N_E(D_1)|/|G : E| = |E : N_E(D_1)|$ which is exactly the number of conjugates of D_1 in E. Therefore by (4.2.1) D_1 and D_2 belong to the same conjugacy class of E as claimed.

Alperin shows in [1] that when $\mathfrak{F} = \mathfrak{N}$ Theorem 6.4.6 is true without the restriction $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$. It would be of interest to know whether this hypothesis can be dropped for general $\mathfrak{F}$.

<u>6.4.7</u> <u>THEOREM</u>. Let $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$ and let H be a minimal member of a chain of the form

$$H = H_r < H_{r-1} < \dots < H_1 < H_0 = G,$$

where H_i is a maximal subgroup of H_{i-1} f-abnormal for $L \cap H_{i-1}$, $i = 1, 2, \dots, r$. Then $L \cap H \in \mathfrak{F}$, and L has an $\mathfrak{F}$ - normalizer D and an $\mathfrak{F}$ - covering subgroup E relative to G such that $D \leq H \leq E$.

<u>Proof</u>. Since H is minimal, every maximal subgroup of H is f-normal for $L \cap H$, and therefore by (2.4.7) $L \cap H \in \mathfrak{F}$. Moreover, the existence of a D contained in H follows at once from (4.4.1), even without the restriction $L \in \mathfrak{N}\mathfrak{N}\mathfrak{F}$. Therefore it remains to show the existence of E, and we do this by induction

on the length of the chain. The result is true when $r = 0$; for then $H = H_0 = G$, $L \in \mathfrak{F}$ and G is itself an $\mathfrak{F}$ - covering subgroup of L relative to G . Therefore assume $r \geq 1$ so that by induction H is contained in an $\mathfrak{F}$ - covering subgroup E^* of $L \cap H_1$ relative to H_1 . Write $K_1 = F(L)$ and $K_2/K_1 = F(L/K_1)$. Since by hypothesis $L/K_2 \in \mathfrak{F}$, by the f-abnormality of H_1 for L we have $H_1 K_2 = G$. If $H_1 \geq K_1$, since $K_2/K_1 \in \mathfrak{N}$, by (2.3.8) $K_2/H_1 \cap K_2$ is a chief factor of G and by (2.3.6) it is evidently $\mathfrak{F}$ -crucial for L . In this case, therefore, H_1 is $\mathfrak{F}$ -crucial for L and E^* is an $\mathfrak{F}$ - covering subgroup of L relative to G . On the other hand, if $H_1 \ngeq K_1$, then H_1 supplements K_1 in G , and by (6.2.3) we have $E^* = E \cap H_1$ for suitable $\mathfrak{F}$ - covering subgroup E of L relative to G . This completes the proof.

As observed by Alperin in [1] this theorem is not true for $L \in \mathfrak{N}\mathfrak{N}\mathfrak{N}\mathfrak{F}$; for the group G in Example 5.3.5 has an abnormal maximal subgroup M whose Carter subgroups are not contained in Carter subgroups of G .

6.5 We end this chapter with a few words about the covering and avoidance property of relative $\mathfrak{F}$ - normalizers.

6.5.1 THEOREM. Let $L \in \mathfrak{N}\mathfrak{F}$, and let X be a subgroup of G which covers all those chief factors in a given chief series of G which are f-central for L ; then X contains an $\mathfrak{F}$ - normalizer

D of L relative to G. If, in addition, X avoids those chief factors in the given chief series which are f-eccentric for L, then $X = D$.

Proof. We use induction on $|G|$. Let N be the minimal normal subgroup in the given chief series. Since $|G/N| < |G|$ and the hypotheses carry over to this factor group, by induction we have $XN/N \geq D^*/N$ for some $\mathfrak{F}$ - normalizer D^*/N of LN/N relative to G/N. By (4.2.4) $D^* = DN$ for a suitable $\mathfrak{F}$ - normalizer D of L relative to G. By a well-known theorem $F(L) \leq C_G(N)$, and since $D\,F(L) = G$, we have $Aut_D(N) \cong Aut_G(N)$. Thus N is a chief factor of D^*, and therefore a fortiori a chief factor of $X^* = XN \geq DN$. If $N \leq X$, there is nothing further to prove. If $N \nleq X$ on the other hand, X avoids N, N is therefore f-eccentric for $L = (L \cap X^*)\,F(L)$ and hence also for $L \cap X^*$. Hence X is a maximal subgroup of X^* f-critical for $L \cap X^*$, and therefore by (4.4.4) X contains an $\mathfrak{F}$ - normalizer D^* of $L \cap X^*$ relative to X^*. But $L \cap X^* \in \mathfrak{N}\mathfrak{F}$ and so by (5.5.1) D^* is an $\mathfrak{F}$ - covering subgroup of $L \cap X^*$ relative to X^*. Again by (5.5.1) D is an $\mathfrak{F}$ - covering subgroup of L relative to G, and therefore by (5.2.4) D is an $\mathfrak{F}$ - covering subgroup of $L \cap X^*$ relative to X^*. By (5.2.2) D is therefore conjugate to D^* so that D^* is also an $\mathfrak{F}$ - normalizer of L relative to G. This proves the first statement and the second follows at once from (4.2.2) and considerations of order.

It is well-known that if $G \in \mathfrak{N}^3$ a subgroup of G which covers all the central chief factors and avoids all the eccentric chief factors need not be a system normalizer of G. We now give an example to show that the supersoluble normalizers of an $\mathfrak{N}^3$-group may also fail to be characterized by the covering and avoidance property.

6.5.2 EXAMPLE. Let B and C be elementary Abelian groups of order 5^3 defined by

$$B = \langle b_1, b_2, b_3, b_4 \mid b_i^5 = 1 , [b_i , b_j] = 1 , b_1 b_2 b_3 b_4 = 1 \rangle$$

and C likewise. Let $Z = \langle z \rangle$ be a cyclic group of order 5, and define a group A of automorphisms of $Z \times B \times C$ by four generators a_i satisfying the following relations:

$$\begin{aligned} z^{a_i} &= z\, b_i\, c_i \ ; \\ b_i^{a_i} &= b_i\, c_i^3 \ ; \\ b_j^{a_i} &= b_j\, c_i\, c_j \ , (i \neq j) \ ; \\ c_j^{a_i} &= c_j \ ; \end{aligned}$$

where $i, j = 1, 2, 3, 4$. It is not difficult to verify that A is an elementary Abelian group of order 5^3 whose four generators satisfy the same relations as those of B. Let K be the splitting extension of $Z \times B \times C$ by A. Now for each permutation $\sigma \in A_4$, the alternating group of degree 4, define a corresponding mapping s of K into itself by its effect on the generators of K as follows:

$$z^s = z \ ; \quad a_i^{\,s} = a_{i\sigma}; \quad b_i^{\,s} = b_{i\sigma} \ ; \quad c_i^{\,s} = c_{i\sigma} \ .$$

It is easy to see that the set of all such s forms a group R of automorphisms of K isomorphic with A_4. Let G be the splitting extension of K by R; then G is a Sylow tower group (and therefore has p-length one for all primes p) and has nilpotent length 3 and order $2^2.3.5^{10}$. Let T be the normal subgroup of R of order 4; then G has a chief series $1 < C < B \times C < Z \times B \times C < K < KT < G$. In this series G/KT and $Z \times B \times C/B \times C$ are the only cyclic chief factors and both are central; therefore the $\mathfrak{U}$ - normalizers of G coincide with its system normalizers and have order 15. Let $s \in R$ correspond to the cycle $(123) \in A_4$. Then $S = \langle s \rangle \times \langle z \rangle$ is a $\mathfrak{U}$-normalizer of G. Since the $\mathfrak{U}$- normalizers form a conjugacy class of $G = KR$, and because R centralizes $\langle z \rangle$, the Sylow 5-subgroups of the $\mathfrak{U}$ - normalizers are the conjugates of $\langle z \rangle$ in K. But the defining relations show that $\langle zc_4 \rangle$ is not a conjugate of $\langle z \rangle$ in K, and therefore the subgroup $S^* = \langle s \rangle \times \langle zc_4 \rangle$ is not a $\mathfrak{U}$ - normalizer of G. However, since C is the unique minimal normal subgroup of G, S^* covers every chief factor covered by S. Therefore S^*, although not a $\mathfrak{U}$- normalizer of G, covers every cyclic chief factor and avoids every non-cyclic chief factor of G.

Chapter Seven

A THEOREM ON THE NILPOTENT ϖ-LENGTH OF A SOLUBLE GROUP

7.1 The concept of the nilpotent ϖ-length of a soluble group discussed in this chapter is a generalization of the well-known concept of nilpotent length as well as of the concept of p-length investigated by P. Hall and Higman in [15]. For completeness we discuss the elementary facts at length in this first section. The results are clearly applicable to groups which are p-soluble for all $p \in \varpi$ (in the sense of Hall and Higman, loc. cit.) ; however, we do not need such generality here since all groups considered in this chapter are soluble.

7.1.1 DEFINITIONS. We say a normal series $1 = G_0 < G_1 < \dots < G_r = G$ is a nilpotent ϖ-series (written $\mathfrak{N}_\varpi$-series) of G if either

(a) $G_i/G_{i-1} \in \mathfrak{N} \cap \mathfrak{R}_\varpi = \mathfrak{N}_\varpi$, or

(b) $G_i/G_{i-1} \in \mathfrak{R}_{\varpi'}$,

for $i = 1, 2, \dots, r$. We call the least number of $\mathfrak{N}_\varpi$-factors occuring in any $\mathfrak{N}_\varpi$-series of G the nilpotent ϖ-length of G and write it $l_\varpi(G)$. If ϖ is a single prime p, then $l_\varpi(G)$ is the p-length of G, and if ϖ is the set of all primes $l_\varpi(G)$ is the nilpotent length $l(G)$ of G. We define the upper $\mathfrak{N}_\varpi$-series of G inductively as follows: $P_0 = 1$; N_i/P_{i-1} is the largest normal ϖ'-subgroup of G/P_{i-1} ; and P_i/N_i is the largest nilpotent normal ϖ-subgroup of G/N_i , for $i = 1, 2, \ldots$. This definition is permissible since both the classes $\mathfrak{R}_{\varpi'}$ and $\mathfrak{N}_{\varpi}$ are N_0-closed. We remark that whereas it is possible to have $P_{i-1} = N_i$, the group P_i/N_i is always non-trivial by the assumption that G is soluble. Since $\mathfrak{R}_{\varpi'}$ and $\mathfrak{N}_{\varpi}$ are formations we may likewise define the <u>lower $\mathfrak{N}_{\varpi}$-series</u> of G inductively as follows: $P_0^* = G$, N_i^* is the $\mathfrak{R}_{\varpi'}$-residual of P^*_{i-1} ; P_i^* is the $\mathfrak{N}_{\varpi}$-residual of N_i^* , for $i = 1, 2, \ldots$.

<u>7.1.2</u> <u>LEMMA</u>. Let G have upper $\mathfrak{N}_{\varpi}$-series $1 = P_0 \leq N_1 < P_1 \leq N_2 < \ldots$, and lower $\mathfrak{N}_{\varpi}$-series $G = P_0^* \geq N_1^* > P_1^* \geq N_2^* > \ldots$. If $1 = G_0 \leq G_1 \leq \ldots \leq G_{2r} \leq G_{2r+1} = G$ is any $\mathfrak{N}_{\varpi}$-series in which G_{2i}/G_{2i-1} is an $\mathfrak{N}_{\varpi}$-group for $i = 1, 2, \ldots, r$ and G_{2i+1}/G_{2i} is a ϖ'-group for $i = 0, 1, \ldots, r$, then

(a) $G_{2i} \leq P_i$, $G_{2i+1} \leq N_{i+1}$ for $i = 0, 1, \ldots, r$, and

(b) $G_{2i} \geq N^*_{r-i+1}$, $G_{2i+1} \geq P^*_{r-i}$ for $i = 0, 1, \ldots, r$.

<u>Proof</u>. (a) We have $G_0 = P_0$, and since G_1 is a normal ϖ'-subgroup of G we have $G_1 \leq N_1$ by the N_0-closure of $\mathfrak{R}_{\varpi'}$. Since the statement is therefore true for $i = 0$, we use induction on i . Assume the result has already been proved for all integers less than i so that we may assume $G_{2i-1} \leq N_i$. By hypothesis $G_{2i}/G_{2i-1} \in \mathfrak{N}_{\varpi}$ and hence $G_{2i}N_i/N_i \cong G_{2i}/G_{2i} \cap N_i \in Q(G_{2i}/G_{2i-1}) \leq Q\mathfrak{N}_{\varpi} = \mathfrak{N}_{\varpi}$. Therefore $G_{2i}N_i/N_i$, as a normal $\mathfrak{N}_{\varpi}$-subgroup of G/N_i , is contained in P_i/N_i since $\mathfrak{N}_{\varpi}$ is N_0-closed. Similarly $G_{2i+1}P_i/P_i \cong G_{2i+1}/G_{2i+1} \cap P_i \in Q(G_{2i+1}/G_{2i}) \leq \mathfrak{R}_{\varpi'}$, and by the N_0-closure of

$\mathfrak{R}_{\varpi'}$ we have $G_{2i+1}P_i/P_i \leq N_{i+1}/P_i$. This completes the induction step. The proof of (b) is similar to that of (a), depending on the $\{S,R_0\}$-closure of both $\mathfrak{R}_{\varpi'}$ and $\mathfrak{N}_{\varpi}$, and will be omitted.

7.1.3 COROLLARY. The integer $l_{\varpi}(G)$ is equal to the number of $\mathfrak{N}_{\varpi}$-factors in both the upper and the lower $\mathfrak{N}_{\varpi}$-series of G .

Proof. Let $n = l_{\varpi}(G)$. Then it is evident that an $\mathfrak{N}_{\varpi}$-series containing the least number of $\mathfrak{N}_{\varpi}$-factors may be written in the form $1 = G_0 \leq G_1 \leq \ldots \leq G_{2r} \leq G_{2r+1} = G$ described above in the statement of (7.1.2) with $r = n$, by amalgamating if necessary any consecutive ϖ'-factors. By (7.1.2 (a)) we have $N_{n+1} \geq G_{2n+1} = G$, and hence $N_{n+1} = G$. But if $N_n = G$, we should have an $\mathfrak{N}_{\varpi}$-series of G with $n - 1 < l_{\varpi}(G)$ $\mathfrak{N}_{\varpi}$-factors, contradicting the definition of $l_{\varpi}(G)$. Hence $N_n < G$ and so the upper $\mathfrak{N}_{\varpi}$-series has precisely $n = l_{\varpi}(G)$ $\mathfrak{N}_{\varpi}$-factors. A similar argument applied to the lower $\mathfrak{N}_{\varpi}$-series shows that $N^*_{n+1} = 1$ but $N^*_n \neq 1$.

7.1.4 LEMMA. The following subgroups of G are identical:

(a) The second term $O_{\varpi'\underline{\varpi}}(G)$ of the upper $\mathfrak{N}_{\varpi}$-series of G ;

(b) The intersection of the $\mathfrak{N}^p$-radicals $O_{p'p}(G)$ as p runs through ϖ ;

(c) The intersection of the centralizers of the ϖ-chief factors of G.

Proof. The equality of (b) and (c) is implied by the remarks following (2.4.5). Now let $p \in \varpi$; since $O_{\varpi'\underline{\varpi}}(G)$ is clearly p-nilpotent, we have $O_{\varpi'\underline{\varpi}}(G) \leq O_{p'p}(G)$. However if $q \in \varpi'$, $\bigcap_{p\in\varpi} O_{p'}(G)$ contains all the Sylow q-subgroups of $D = \bigcap_{p\in\varpi} O_{p'p}(G)$. Hence $\sigma(D/O_{\varpi'}(D)) \leq \varpi$, and since $D/O_{\varpi'}(D)$ is p-nilpotent for all $p \in \varpi$

we therefore have $D/O_{\varpi'}(D) \in \mathfrak{N}$. Hence $D \leq O_{\varpi'\varpi}(G)$ and the equality of (a) and (b) is established.

7.1.5 LEMMA. The class $\mathcal{L}_{\varpi}(n)$ comprising all groups G satisfying $l_{\varpi}(G) \leq n$, $(n \geq 1)$, is an S-closed formation defined locally by $f_n(p) = \mathcal{L}_{\varpi}(n-1)$ for $p \in \varpi$ and $f_n(p) = \mathcal{S}$ for $p \in \varpi'$.

Proof. To make sense of the statement of the lemma, we adopt the convention that $\mathcal{L}_{\varpi}(0) = \mathcal{R}_{\varpi'}$, which is certainly an S-closed formation. We assume that $\mathcal{L}_{\varpi}(n-1)$ is an S-closed formation; for if we can prove the lemma under this hypothesis, the conclusion that $\mathcal{L}_{\varpi}(n)$ is also an S-closed formation justifies this assumption by induction. The S-closure of $\mathcal{L}_{\varpi}(n)$ follows at once from the S-closure of f_n by (2.2.2). Now suppose that $G \in \mathcal{L}_{\varpi}(n)$ so that by (7.1.3) $G/O_{\varpi'\varpi}(G) \in \mathcal{L}_{\varpi}(n-1)$; then by (7.1.4), if $p \in \varpi$, we have $G/O_{p'p}(G) \in Q\,\mathcal{L}_{\varpi}(n-1) = \mathcal{L}_{\varpi}(n-1)$, and G belongs to the formation defined locally by f_n. Conversely, suppose $G/O_{p'p}(G) \in f_n(p) = \mathcal{L}_{\varpi}(n-1)$ for all $p \in \varpi$. By assumption $\mathcal{L}_{\varpi}(n-1) = R_0\,\mathcal{L}_{\varpi}(n-1)$, and therefore by (7.1.4) $G/O_{\varpi'\varpi}(G) = G/\bigcap_{p \in \varpi} O_{p'p}(G) \in \mathcal{L}_{\varpi}(n-1)$. Hence $G \in \mathcal{L}_{\varpi}(n)$ by (7.1.3) and the proof is complete.

We end this section by introducing a second invariant m_{ϖ} which we shall need to carry through the induction arguments in the proof of Theorem 7.2.8 .

7.1.6 DEFINITION. We define the reduced $\mathfrak{N}_{\varpi}$-length $m_{\varpi}(G)$ of G

by $m_{\varpi}(G) = l_{\varpi}(G/O_{\varpi}(G))$. We denote by $\mathfrak{M}_{\varpi}(n)$ the class of groups G which satisfy $m_{\varpi}(G) \leq n$, $n \geq 0$.

7.1.7 LEMMA. The class $\mathfrak{M}_{\varpi}(n)$ is an S-closed formation defined locally by $f_n^*(p) = \mathcal{L}_{\varpi}(n)$ for all primes p , $n \geq 0$.

Proof. Let $G \in \mathfrak{M}_{\varpi}(n)$. For each prime p $O_{\varpi}(G) \leq O_{p'p}(G)$ and therefore we have $G/O_{p'p}(G) \in Q(G/O_{\varpi}(G)) \leq Q\,\mathcal{L}_{\varpi}(n) = \mathcal{L}_{\varpi}(n)$. Hence G belongs to the formation defined by f_n^* . Conversely suppose $G/O_{p'p}(G) \in f_n^*(p) = \mathcal{L}_{\varpi}(n)$ for all primes p . Since by (7.1.5) $\mathcal{L}_{\varpi}(n)$ is a formation $(n \geq 0)$, we have $G/F(G) = G/\bigcap_p O_{p'p}(G) \in \mathcal{L}_{\varpi}(n)$. But $O_{\varpi}(G)$ is the unique Hall ϖ-subgroup of $F(G)$ and therefore $F(G)/O_{\varpi}(G)$ is a normal ϖ'-subgroup of $G/O_{\varpi}(G)$. Hence $l_{\varpi}(G/O_{\varpi}(G)) = l_{\varpi}(G/F(G)) \leq n$ and $G \in \mathfrak{M}_{\varpi}(n)$ as required. The S-closure of $\mathfrak{M}_{\varpi}(n)$ follows from the S-closure of $\mathcal{L}_{\varpi}(n)$ by (2.2.2) .

7.2 In unpublished work M.B. Powell has shown by considering certain properties of varieties of groups that every finite soluble group G has a two-generator subgroup H such that $l_p(H) = l_p(G)$. In a similar vein an unpublished result due to R.W. Carter and B. Fischer says that for any local formation $\mathfrak{F}$, G always possesses $\mathfrak{F}$-covering subgroups E_1 and E_2 such that the subgroup $H = \langle E_1, E_2 \rangle$ satisfies $\sigma(H) = \sigma(G)$ and $l(H) = l(G)$. They asked whether this result would remain true if "l" were replaced by the p-length function "l_p" . We answer this question in the affirmative, and show that all these results are in fact consequences of a more general theorem. We prove the theorem in this section and discuss the consequences in section 7.3 . First, however, we need some definitions and preliminary lemmas.

7.2.1 DEFINITION. We call a class $\mathfrak{X}$ of groups extreme if it is Q-closed and if whenever a group G has a minimal normal subgroup N such that $G/N \in \mathfrak{X}$ and such that either (i) $N \leq \Phi(G)$, or (ii) N is complemented in G and all complements are conjugate, then $G \in \mathfrak{X}$. If $\mathfrak{X}$ is extreme it follows that if $G/\Phi(G) \in \mathfrak{X}$ then $G \in \mathfrak{X}$; this is an easy consequence of condition (i). From condition (ii) it follows that $\mathfrak{X}$ contains the groups of order a prime.

7.2.2 DEFINITION. If for each $G \neq 1$ is specified (to within isomorphism class of G) a non-empty set $P(G)$ of non-trivial $\mathfrak{X}$-subgroups of G such that

(i) $G \in \mathfrak{X} \Rightarrow G \in P(G)$,

(ii) $H \leq L$ and $H \in P(G) \Rightarrow P(L) \leq P(G)$, and

(iii) for $K \triangleleft G$ we have $P(G/K) = \{ HK/K \mid H \in P(G), H \nleq K \}$,

then we say P is a permissive function (for $\mathfrak{X}$) and we call the members of $P(G)$ the permissible subgroups of G (with respect to P). We shall show in section 7.3 that the class $\mathfrak{G}_r$ $(r \geq 2)$ of groups generated by at most r elements is an extreme class, and as a simple example of a permissive function to illustrate the theory we could define $P(G)$ to be the set of all non-trivial $\mathfrak{G}_r$-subgroups of G . In fact Theorem 7.2.4 below shows that there is at least one permissive function for each extreme class. For the proof of (7.2.4) we need

7.2.3 LEMMA. Let $\mathfrak{X}$ be an extreme class, and K a normal subgroup of G with $G/K \in \mathfrak{X}$. If H is a minimal member of the set $\underline{S}$ of subgroups supplementing K in G, then $H \in \mathfrak{X}$.

Proof. We first remark that $\underline{S}$ is non-empty for $\underline{S}$ contains G. $\underline{S}$ therefore has minimal members H. Now $H/H \cap K \cong HK/K = G/K \in \mathfrak{X}$. Let H^* be a maximal subgroup of H supplementing $H \cap K$ in H; then $H^*K = H^*(H \cap K)K = HK = G$. But this is impossible by the minimality of H. Hence every maximal subgroup of H contains $H \cap K$ and therefore $H \cap K \leq \phi(H)$. Hence $H/\phi(H) \in Q\mathfrak{X} = \mathfrak{X}$, and therefore by the remarks following (7.2.1) we have $H \in \mathfrak{X}$.

7.2.4 THEOREM. If $\mathfrak{X}$ is an extreme class, the function P defined by

$$P(G) = \{ X \mid 1 \neq X \leq G , \ X \in \mathfrak{X} \}$$

is a permissive function.

Proof. If $G \neq 1$, G has a subgroup of order a prime which by the remarks following (7.2.1) is an $\mathfrak{X}$-subgroup. Hence $P(G)$ is non-empty. Moreover P obviously fulfils requirements (i) and (ii) of (7.2.2), and so it remains to show P satisfies (iii). Let $\underline{T}$ denote the set $\{ HK/K \mid H \in P(G) , \ H \nleq K \}$. A typical element of $\underline{T}$ satisfies $1 \neq HK/K = H/H \cap K \in Q\mathfrak{X} = \mathfrak{X}$, and therefore $\underline{T} \leq P(G/K)$. Now suppose $H^*/K \in P(G/K)$. We have $H^*/K \in \mathfrak{X}$, and therefore by (7.2.3) H^* has an $\mathfrak{X}$-subgroup H such that $HK = H^*$. Moreover, since $H^*/K \neq 1$, we have $H \nleq K$ and therefore $P(G/K) \leq \underline{T}$. Hence $P(G/K) = \underline{T}$ and the proof is complete.

The next lemma is a simple consequence of Definition 7.2.2.

7.2.5 LEMMA. If $K \triangleleft G$ and $H/K \in P(G/K)$, then $P(H) \leq P(G)$.

Proof. By condition (iii) of (7.2.2) we have $H/K = LK/K$ for some $L \in P(G)$, and the conclusion follows from condition (ii).

7.2.6 LEMMA. Let $\mathfrak{X}$ be an extreme class and P a permissive function for $\mathfrak{X}$. If G has a minimal normal subgroup N such that $G/N \in \mathfrak{X}$ but $G \notin \mathfrak{X}$, then G has a subgroup H complementing N such that $H \in P(G)$ and $\sigma(H) = \sigma(G)$.

Proof. By condition (i) of (7.2.2) $G/N \in P(G/N)$, and therefore by condition (iii) $G/N = HN/N$ for some $H \in P(G)$. Since $G \notin \mathfrak{X}$, $G \notin P(G)$, and therefore $H \neq G$; hence H must be a complement of N in G. If $\sigma(H) \neq \sigma(G)$, N must be a Sylow p-subgroup of G. But then since all Sylow p-complements of G are conjugate, by requirement (ii) of (7.2.1) we should have $G \in \mathfrak{X}$ contrary to hypothesis. Hence $\sigma(H) = \sigma(G)$ and the proof is complete.

For convenient reference we now assemble some well-known facts concerning a situation encountered in the earlier chapters (see for example the proof of (5.5.3)).

7.2.7 LEMMA. Let $\mathfrak{F}$ be a saturated formation, and suppose that $G/N \in \mathfrak{F}$ for every minimal normal subgroup N of G, but that $G \notin \mathfrak{F}$. Then G has a unique minimal normal subgroup, N^* say, which is complemented in G and all of whose complements are conjugate; moreover $N^* = C_G(N^*) = F(G)$. In particular, it follows from the conjugacy of the complements that if G/N^* belongs to an extreme class $\mathfrak{X}$, then G itself belongs to $\mathfrak{X}$.

7.2.8 THEOREM. Let $\mathfrak{X}$ be an extreme class of groups and P a permissive function for $\mathfrak{X}$. Then every non-trivial finite soluble group G has a subgroup L satisfying the following conditions:

(i) $L \in P(G)$;

(ii) $l_{\varpi}(L) = l_{\varpi}(G)$;

(iii) $m_{\varpi}(L) = m_{\varpi}(G)$;

(iv) $\sigma(L) = \sigma(G)$.

Proof. We deal first with the case $l_{\varpi}(G) = 0$; that is when G is a ϖ'-group. If G is cyclic of prime order, then by the remark following (7.2.1) $G \in \mathfrak{X}$, and hence by condition (i) of (7.2.2) we have $G \in P(G)$. We may therefore assume that G is not of prime order, and that the theorem has been proved already for groups of order less than $|G|$. Let N be a minimal normal subgroup of G so that by assumption G/N is non-trivial. By induction G/N has a permissible subgroup H/N such that $\sigma(H/N) = \sigma(G/N)$, and therefore $\sigma(H) = \sigma(G)$. If $|H| < |G|$ the result follows by induction since by (7.2.5) $P(H) \leq P(G)$. On the other hand, if $H = G$, then $G/N \in \mathfrak{X}$, and either $G \in \mathfrak{X}$ or by (7.2.6) G has a permissible subgroup L such that $\sigma(L) = \sigma(G)$; in either case the theorem is true. Therefore we may assume $l_{\varpi}(G) = n > 0$, and proceed again by induction on $|G|$. We distinguish two situations:

Case 1. $m_{\varpi}(G) = n$. First suppose that G has a minimal normal subgroup N such that $m_{\varpi}(G/N) = n$. A fortiori $l_{\varpi}(G/N) = n$, and therefore by induction G/N has a permissible subgroup H/N such that $l_{\varpi}(H/N) = m_{\varpi}(H/N) = n$, and $\sigma(H/N) = \sigma(G/N)$. If $|H| < |G|$, by induction H has a subgroup L such that $L \in P(H)$, $l_{\varpi}(L) = m_{\varpi}(L) = n$ and $\sigma(L) = \sigma(H)$ $(= \sigma(G))$; by (7.2.5) $P(H) \leq P(G)$ and therefore L fulfils requirements (i) - (iv) of the theorem. On the other hand, if $H = G$, then $G/N \in \mathfrak{X}$, and if $G \notin \mathfrak{X}$ G has a permissible subgroup L such that $\sigma(L) = \sigma(G)$ by (7.2.6) ; in this case $L \cong G/N$ and again L satisfies conditions (i) - (iv). Hence we are left with the case where $m_{\varpi}(G/N) = n-1$ for every minimal normal subgroup N of G . Since by (7.1.7) $\mathfrak{M}_{\varpi}(n-1)$ is a saturated formation for $n \geq 1$, (7.2.7) applies and G has a unique minimal normal subgroup N^* such that $N^* = C_G(N^*) = F(G)$. Since $m_{\varpi}(G) = n$, N^* must be a ϖ'-subgroup; for if N^* were a ϖ-subgroup we should have $N^* = O_{\varpi'\varpi}(G) = O_{\varpi}(G)$ and $m_{\varpi}(G) = l_{\varpi}(G/O_{\varpi}(G)) = l(G/O_{\varpi'\varpi}(G)) = l_{\varpi}(G) - 1 = n-1$. Therefore $l_{\varpi}(G/N^*) = n$. By induction G/N^* has a permissible subgroup H/N^* such that $l_{\varpi}(H/N^*) = m_{\varpi}(H/N^*) + 1 = n$ and $\sigma(H/N^*) = \sigma(G/N^*)$. Therefore $\sigma(H) = \sigma(G)$. Moreover, since $O_{\varpi}(H)$ centralizes $N^* = C_H(N^*)$, we must have $O_{\varpi}(H) = 1$ and therefore $m_{\varpi}(H) = n$. Thus if $|H| < |G|$, by induction H has a permissible subgroup L such that $l_{\varpi}(L) = m_{\varpi}(L) = n$ and $\sigma(L) = \sigma(H)$. By (7.2.5) $P(H) \leq P(G)$ and so L is a permissible subgroup of G with the required properties. If, on the other hand, $H = G$, then by the final remark of (7.2.7) $G \in \mathfrak{X}$ and by definition of P we have $G \in P(G)$. This concludes the proof for case 1.

Case 2. $m_{\varpi}(G) = n-1$. In this case $O_{\varpi}(G)$ must be non-trivial. If G has a minimal normal subgroup N such that $l_{\varpi}(G/N) = n$ then

$m_{\varpi}(G/N) = n-1$ and by induction G/N has a permissible subgroup H/N such that $l_{\varpi}(H/N) = m_{\varpi}(H/N) + 1 = n$ and $\sigma(H/N) = \sigma(G/N)$. Therefore $\sigma(H) = \sigma(G)$ and by (7.2.5) $P(H) \leq P(G)$. If $|H| < |G|$ the result follows by induction by the same argument as before. Otherwise we have $H = G$ and $G/N \in \mathfrak{X}$. In this case either $G \in \mathfrak{X}$ whence $G \in P(G)$, or N has a complement $L \in P(G)$ with $\sigma(L) = \sigma(G)$ by (7.2.6); since $L \cong G/N$, L satisfies requirements (i) - (iv) of the theorem. Hence we may assume $G/N \in \mathcal{L}_{\varpi}(n-1)$ for each minimal normal subgroup N of G. Since $\mathfrak{N}_{\varpi'}$ is well-known to be a saturated formation, for all $n \geq 1$ $\mathcal{L}_{\varpi}(n-1)$ is a saturated formation by (7.1.5); therefore by (7.2.7) G has a unique minimal normal subgroup N^* which is complemented in G, all of whose complements are conjugate and which satisfies $N^* = C_G(N^*) = F(G)$. Since $O_{\underline{\varpi}}(G) \neq 1$, N^* must be a p-group for some $p \in \varpi$. First we consider the case $m_{\varpi}(G/N^*) = n-1$ for this includes the case $l_{\varpi}(G) = 1$. By induction G/N^* has a permissible subgroup H/N^* such that $\sigma(H/N^*) = \sigma(G/N^*)$ and $m_{\varpi}(H/N^*) = n-1$. Thus $\sigma(H) = \sigma(G)$, $m_{\varpi}(H) = n-1$ and by (7.2.5) $P(H) \leq P(G)$. Moreover since $N^* = C_G(N^*)$ we have $O_{\varpi'}(H) = 1$, and therefore $O_{\varpi'\underline{\varpi}}(H) = O_{\underline{\varpi}}(H)$. Hence $l_{\varpi}(H) = l_{\varpi}(H/O_{\varpi'\underline{\varpi}}(H)) + 1 = l_{\varpi}(H/O_{\underline{\varpi}}(H)) + 1 = m_{\varpi}(H) + 1 = n$. If $|H| < |G|$ the result follows by induction as before. If $H = G$, then $G/N^* \in \mathfrak{X}$ and by the final remark of (7.2.7) $G \in \mathfrak{X}$; hence $G \in P(G)$ and the theorem is true. Finally therefore we may assume $n \geq 2$ and $m_{\varpi}(G/N^*) = n-2$. Let R/N^* denote $O_{\underline{\varpi}}(G/N^*)$. Since $N^* = F(G)$ we have $p \nmid |R : N^*|$, and therefore N^* is a Sylow p-subgroup of R. By induction G/N^*

has a permissible subgroup H/N^* such that $l_{\varpi}(H/N^*) = m_{\varpi}(H/N^*) + 1 =$ $n-1$ and $\sigma(H/N^*) = \sigma(G/N^*)$. Hence $\sigma(H) = \sigma(G)$ and $l_{\varpi}(H/H \cap R) =$ $l_{\varpi}(HR/R) \leq n-2$. Moreover, by (7.2.5) $P(H) \leq P(G)$. Let S denote $O_{\varpi}(H)$; since $N^* = C_H(N^*)$, S must be a p-group and therefore $R \cap S =$ N^*. Suppose, for a contradiction, that $m_{\varpi}(H) = n-2$. Then, since $\mathcal{L}_{\varpi}(n-2)$ is a formation, we have $n-2 \geq l_{\varpi}(H/H \cap R \cap S) = l\ (H/N^*) =$ $n-1$, which is impossible. Hence $m_{\varpi}(H) = n-1 = m_{\varpi}(G)$. Further we have $O_{\varpi'}(H) \leq C_H(N^*) = N^*$ and therefore $O_{\varpi'}(H) = 1$. Hence $l_{\varpi}(H) =$ $m_{\varpi}(H) + 1 = n$. If $|H| < |G|$, the result follows by induction as before. If $H = G$, we have $G/N^* \in \mathfrak{X}$ and by (7.2.7) $G \in \mathfrak{X}$. G then belongs to $P(G)$ and fulfils the requirements of the theorem. This completes the proof.

7.3 To deduce Powell's Theorem from (7.2.8) we need the following Lemma which is due to Powell but which was also proved independently by W. Gaschütz. We give Gaschütz's unpublished proof here.

7.3.1 LEMMA. Let N be a minimal normal subgroup of G which is complemented in G and all of whose complements are conjugate, and let r be an integer ≥ 2. Then if $G/N \in \mathcal{G}_r$, the class of groups generated by at most r elements, we also have $G \in \mathcal{G}_r$.

Proof. Let M be a complement of N in G; since $M \cong G/N$, we may write $M = \langle g_1, \ldots, g_r \rangle$. Let $|N| = s$ and enumerate the distinct elements of N thus: $N = \{ n_1 = 1, n_2, \ldots, n_s \}$. Consider the subgroups $M_\mu = \langle g_1 n_{\mu(1)}, \ldots, g_r n_{\mu(r)} \rangle$ as μ runs through the set $\mathfrak{M}$ of s^r mappings of the set $\{ 1, 2, \ldots, r \}$ into the set

$\{ 1, 2, \dots , s \}$. Then for any $\mu \in \mathfrak{M}$ we have $M_\mu N = G$. Suppose, for a contradiction that $M_\mu \neq G$ for all $\mu \in \mathfrak{M}$, and therefore that each M_μ is a complement of N in G . By hypothesis there are $|G : N_G(M)| \leq s$ complements of N in G , but since $r \geq 2$, there are more than s M_μ's . Therefore $M_\mu = M_{\mu'}$ for some $\mu \neq \mu'$. Let i be an integer such that $\mu(i) \neq \mu'(i)$. Now $g_i n_{\mu'(i)} \in M_{\mu'} = M_\mu \ni g_i n_{\mu(i)}$; hence $1 \neq n^{-1}_{\mu'(i)} n_{\mu(i)} = (g_i n_{\mu'(i)})^{-1}(g_i n_{\mu(i)}) \in M_\mu$ and therefore M_μ contains a non-trivial member of N , contradicting our assumption that M_μ complements N . This contradiction proves that $G = M_\mu$ for some $\mu \in \mathfrak{M}$, and therefore that $G \in \mathcal{G}_r$ as required.

<u>7.3.2</u> <u>THEOREM</u>. $\mathcal{G}_r$ is an extreme class, $r \geq 2$.

<u>Proof</u>. Since a set of generators of a group maps under any homomorphism into a set of generators of the image, the class $\mathcal{G}_r$ is Q-closed. Let N be a minimal normal subgroup of G and suppose $G/N \in \mathcal{G}_r$. If $N \leq \Phi(G)$, let $G/N = \langle g_1 N, \dots , g_r N \rangle$ and write $M = \langle g_1, \dots , g_r \rangle$. Then we have $M\Phi(G) \geq MN = G$; this implies $M = G$ and therefore $G \in \mathcal{G}_r$. On the other hand, if N is complemented in G and all its complements are conjugate we can apply (7.3.1) and again deduce that $G \in \mathcal{G}_r$. This proves the theorem.

If $G \neq 1$ define $P(G)$ to be the set of non-trivial 2-generator subgroups of G . By taking $\mathfrak{X} = \mathcal{G}_2$ in (7.2.4) it follows that P is a permissive function for $\mathcal{G}_2$, and therefore applying this to (7.2.8) we get the following specialization.

<u>7.3.3</u> <u>THEOREM</u>. A non-trivial finite soluble group G has a 2-generator subgroup L satisfying (a) $l_{\varpi}(L) = l_{\varpi}(G)$, (b) $m_{\varpi}(L) = m_{\varpi}(G)$ and (c) $\sigma(L) = \sigma(G)$.

If we take ϖ to be a single prime p in this theorem, conclusion (a) yields Powell's Theorem. We now show how to deduce the Carter-Fischer Theorem.

<u>7.3.4</u> <u>DEFINITION</u>. If $\mathfrak{F}$ is a local formation we say a group G is <u>on r $\mathfrak{F}$-generators</u> if $G = \langle E_1, E_2, \ldots, E_r \rangle$ for suitable $\mathfrak{F}$ - covering subgroups E_i of G . Denote by $\mathfrak{E}_r$ the class of groups on r $\mathfrak{F}$-generators.

<u>7.3.5</u> <u>LEMMA</u>. If G has a minimal normal subgroup N complemented in G by H and such that all complements of N in G are conjugate, and if H is on r $\mathfrak{F}$-generators with $r \geq 2$, then G is also on r $\mathfrak{F}$-generators.

<u>Proof</u>. Let $H = \langle E_1^*, E_2^*, \ldots, E_r^* \rangle$ where E_i^* is an $\mathfrak{F}$- covering subgroup of H , $i = 1, 2, \ldots, r$. Since H supplements $F(G)$, by (6.2.3) we have $E_i^* = E_i \cap H$ for a suitable $\mathfrak{F}$- covering subgroup E_i of G , $i = 1, 2, \ldots, r$. Write $L = \langle E_1, E_2, \ldots, E_r \rangle$. If $L = G$ the result is true and so we assume $L < G$. In this case, since $H \leq L$ and $H \lessdot G$, we must have $H = L$, and hence $E_i = E_i^*$, $i = 1, 2, \ldots, r$. By (5.3.3) $E_1 \rtimes G$, and therefore $H = N_G(H)$. Let $1 \neq g \in N$ and write $H^* = \langle E_1, E_2, \ldots, E_{r-1}, E_r^{g} \rangle$. If $H = H^*$, we have $E_r \leq H \cap H^{g^{-1}}$; but $H \neq H^{g^{-1}}$, and since the

abnormal subgroup E_r cannot be contained in two distinct conjugates we have a contradiction. Hence $H \neq H^*$. But $H^*N \geq HN = G$, and therefore either $H^* = G$, or H^* is a complement of N in G. If this second alternative obtains then by hypothesis H and H^* are distinct conjugate subgroups of G; but this is impossible, for as $r \geq 2$ H and H^* then contain the abnormal subgroup E_1 of G. Hence $H^* = G$ and therefore G is on r $\mathfrak{F}$-generators as required.

7.3.6 THEOREM. $\mathcal{E}_r$ is an extreme class, $r \geq 2$.

Proof. The Q-closure of $\mathcal{E}_r$ follows at once from (5.2.6). Let N be a minimal normal subgroup of G and suppose $G/N \in \mathcal{E}_r$. By (5.2.7) $G/N = < E_1N/N , \ldots , E_rN/N >$ for suitable $\mathfrak{F}$- covering subgroups E_i of G. Hence if $N \leq \phi(G)$ the subgroup $M = < E_1, E_2, \ldots , E_r >$ supplements $\phi(G)$ whence $M = G$ and therefore $G \in \mathcal{E}_r$. If, on the other hand, N is complemented and all complements are conjugate then (7.3.5) applies, and we again conclude that $G \in \mathcal{E}_r$. Thus $\mathcal{E}_r$ fulfils the requirements of (7.2.1) and is therefore an extreme class.

7.3.7 LEMMA. The subgroup function P_r defined on $G \neq 1$ by $P_r(G) = \{ H \mid H = < E_1, E_2, \ldots , E_r >$ where the E_i are $\mathfrak{F}$ - covering subgroups of $G \}$, $r \geq 2$, is a permissive function for $\mathcal{E}_r$.

Proof. Let $H \in P_r(G)$. Since the $\mathfrak{F}$- covering subgroups of G are non-trivial, a fortiori $H \neq 1$. Moreover by (5.2.4) E_i is an $\mathfrak{F}$ - covering subgroup of H, $i = 1, 2, \ldots , r$, and therefore $H \in \mathcal{E}_r$. We now show P_r fulfils each of the requirements (i), (ii) and (iii) of Definition 7.2.2 in turn. If $G \in \mathcal{E}_r$ then $G \in P_r(G)$

and (i) is certainly satisfied. If $P_r(G) \ni H \leq L$ then L contains an $\mathfrak{F}$ - covering subgroup of G and by (5.2.4) the $\mathfrak{F}$ - covering subgroups of L are all $\mathfrak{F}$ - covering subgroup of G. Thus $P_r(L) \leq P_r(G)$ and (ii) is satisfied. Finally it follows at once from (5.2.7) that (iii) is satisfied, and therefore P_r is a permissive function as claimed.

By (7.3.6) and (7.3.7) we may take $\mathfrak{X} = \mathfrak{E}_2$ and $P = P_2$ in (7.2.8) to obtain the following special case.

<u>7.3.8</u> <u>THEOREM</u>. A finite soluble group G has $\mathfrak{F}$ - covering subgroups E_1 and E_2 such that the subgroup $L = \langle E_1, E_2 \rangle$ satisfies (a) $m_{\varpi}(L) = m_{\varpi}(G)$, (b) $l_{\varpi}(L) = l_{\varpi}(G)$ and (c) $\sigma(L) = \sigma(G)$.

If we now take ϖ to be the set of all primes, conclusions (b) and (c) of this theorem yield the Theorem of Carter and Fischer.

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